ABELIANIZATION CONJECTURES FOR SOME ARITHMETIC SQUARE COMPLEX GROUPS

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ABSTRACT. We extend a conjecture of Kimberley-Robertson on the abelianizations of certain square complex groups.

1. Introduction

Throughout this paper, let p, l be any pair of distinct odd prime numbers,

$$r_{p,l} := \gcd\left(\frac{p-1}{4}, \frac{l-1}{4}, 6\right) \in \{1,2,3,6\},$$

and $q \in \{p,l\}$. We first recall the definition of the group $\Gamma_{p,l}$ from [2, 3, 4, 5]. Let $\mathbb{Q}_q$ be the field of q-adic numbers. We fix elements $c_p, d_p \in \mathbb{Q}_p$ and $c_l, d_l \in \mathbb{Q}_l$ such that

$$c_p^2 + d_p^2 + 1 = 0 \in \mathbb{Q}_p \text{ and } c_l^2 + d_l^2 + 1 = 0 \in \mathbb{Q}_l.$$

Note that we can take $d_q = 0$, if $q \equiv 1 \pmod 4$.

Let $\mathbb{H}(\mathbb{Q})^*$ be the multiplicative group of invertible rational Hamilton quaternions, i.e. the set

$$\{x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in \mathbb{Q}\} \setminus \{0\}$$

equipped with the multiplication induced by the rules $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$. If $x = x_0 + x_1 i + x_2 j + x_3 k$, we define as usual the conjugate $\overline{x} := x_0 - x_1 i - x_2 j - x_3 k$, and the norm $|x|^2 := x\overline{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2$.

Let ψ_q be the homomorphism of groups $\mathbb{H}(\mathbb{Q})^* \to \mathrm{PGL}_2(\mathbb{Q}_q)$ defined by

$$\psi_q(x_0 + x_1 i + x_2 j + x_3 k) := \left[\begin{pmatrix} x_0 + x_1 c_q + x_3 d_q & -x_1 d_q + x_2 + x_3 c_q \\ -x_1 d_q - x_2 + x_3 c_q & x_0 - x_1 c_q - x_3 d_q \end{pmatrix}\right]$$

and let the homomorphism

$$\psi_{p,l} : \mathbb{H}(\mathbb{Q})^* \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$

be given by $\psi_{p,l}(x) := (\psi_p(x), \psi_l(x))$. Observe that it satisfies $\psi_{p,l}(-x) = \psi_{p,l}(x)$ and $\psi_{p,l}(x)^{-1} = \psi_{p,l}(\overline{x})$.

Let $\mathbb{H}(\mathbb{Z})$ be the set of integer Hamilton quaternions and X_q the subset of quaternions

$$X_q := \{x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})\,;\quad |x|^2 = q\,;$$
$$x_0 \text{ odd, if } q \equiv 1 \pmod 4\,;\ x_1 \text{ even, if } q \equiv 3 \pmod 4\}$$

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of cardinality $2(q+1)$.

Finally, let $Q_{p,l}$ be the subgroup of $\mathbb{H}(\mathbb{Q})^*$ generated by $(X_p \cup X_l) \subset \mathbb{H}(\mathbb{Z})$ and let $\Gamma_{p,l} < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$ be its image $\psi_{p,l}(Q_{p,l})$, which is a finitely presented, torsion-free linear group.

The starting point for this work was the following conjecture of Kimberley and Robertson for the abelianization $\Gamma_{p,l}^{ab} := \Gamma_{p,l}/[\Gamma_{p,l}, \Gamma_{p,l}]$ of the group $\Gamma_{p,l}$ in the case $p, l \equiv 1 \pmod 4$. We use the notation $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_n^m := \mathbb{Z}/n\mathbb{Z} \times \ldots \times \mathbb{Z}/n\mathbb{Z}$ (m times).

Conjecture 1. *(Kimberley-Robertson* [1, Section 6]*) If $p, l \equiv 1 \pmod 4$, then*

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_4^3\,, & \text{if } r_{p,l} = 1 \\ \mathbb{Z}_2^3 \times \mathbb{Z}_8^2\,, & \text{if } r_{p,l} = 2 \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3\,, & \text{if } r_{p,l} = 3 \\ \mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2\,, & \text{if } r_{p,l} = 6\,. \end{cases}$$

In Section 2, we will give an equivalent formulation of this conjecture and a new conjecture relating the abelianization of $\Gamma_{p,l}$ to the number $t_{p,l}$ of certain pairs of commuting quaternions, defined as

$$t_{p,l} := |\{(x, y) \in Y_p \times Y_l \, : \, xy = yx\}|\,,$$

where Y_q is any subset of X_q of cardinality $(q+1)/2$ such that $x \in Y_q$ implies $x_0 > 0$ and $\overline{x} \notin Y_q$. Note that the definition of $t_{p,l}$ does not depend on the choice of elements in Y_p and Y_l, and that $\psi_{p,l}(Y_p \cup Y_l)$ is a generating set of $\Gamma_{p,l}$ of cardinality $(p+1)/2 + (l+1)/2$.

The case $p, l \equiv 3 \pmod 4$ is treated in Section 3 and the remaining (mixed) case in Section 4. The final section is devoted to conjectures on the abelianizations of some subgroups of $\Gamma_{p,l}$.

The Conjectures 2, 3, 4, 5, 7, 9, 10, 11 and 12 have been stated in the authors Ph.D. thesis ([2, Chapter 3]). We have checked Conjectures 2, 4, 5, 7, 9, 10, 11 and 12 for more than 100 different pairs (p, l) which are explicitly listed in [2, Table 3.13].

2. The case $p, l \equiv 1 \pmod 4$

In this section, we restrict to the "classical" case $p, l \equiv 1 \pmod 4$. The following conjecture is equivalent to Conjecture 1.

Conjecture 2. *Let $p, l \equiv 1 \pmod 4$.*
If $p, l \equiv 1 \pmod 8$, then

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2\,, & \text{if } p, l \equiv 1 \pmod 3 \\ \mathbb{Z}_2^3 \times \mathbb{Z}_8^2\,, & \text{else}\,. \end{cases}$$

If $p \equiv 5 \pmod 8$ or $l \equiv 5 \pmod 8$, then

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3\,, & \text{if } p, l \equiv 1 \pmod 3 \\ \mathbb{Z}_2 \times \mathbb{Z}_4^3\,, & \text{else}\,. \end{cases}$$

Proof of the equivalence of Conjecture 1 and Conjecture 2. If $r_{p,l} = 6$, then $(p-1)/4 = 6s$ and $(l-1)/4 = 6t$ for some $s, t \in \mathbb{N}$, i.e. $p = 24s+1$ and $l = 24t+1$. It follows $p, l \equiv 1 \pmod 8$ and $p, l \equiv 1 \pmod 3$.

If $r_{p,l} = 3$, then $(p-1)/4 = 3s$ and $(l-1)/4 = 3t$, where s or t is odd (otherwise $r_{p,l}$ would be 6). Consequently, we have $p = 12s+1$ and $l = 12t+1$, in particular $p, l \equiv 1 \pmod 3$. If s is odd, then $p \equiv 5 \pmod 8$. If t is odd, then $l \equiv 5 \pmod 8$.

If $r_{p,l} = 2$, then $(p-1)/4 = 2s$ and $(l-1)/4 = 2t$, i.e. $p = 8s+1$ and $l = 8t+1$, hence $p, l \equiv 1 \pmod 8$. Moreover, $s \not\equiv 0 \pmod 3$ or $t \not\equiv 0 \pmod 3$ (otherwise $r_{p,l}$ would be 6). In the first case, we have $p \not\equiv 1 \pmod 3$, in the second case $l \not\equiv 1 \pmod 3$.

If $r_{p,l} = 1$, then $(p-1)/4 = 2s-1$ or $(l-1)/4 = 2t-1$ (otherwise $r_{p,l}$ would be even), hence $p = 8s-3$ or $l = 8t-3$, i.e. $p \equiv 5 \pmod 8$ or $l \equiv 5 \pmod 8$. Moreover: $(p-1)/4 = 3s+1$ or $(p-1)/4 = 3s+2$ or $(l-1)/4 = 3t+1$ or $(l-1)/4 = 3s+2$ for some $s, t \in \mathbb{N}_0$ (otherwise $r_{p,l}$ would be a multiple of 3), hence $p = 12s+5$ or $p = 12s+9$ or $l = 12t+5$ or $l = 12t+9$, in particular $p \not\equiv 1 \pmod 3$ or $l \not\equiv 1 \pmod 3$. □

The equivalence of the two conjectures above is also expressed in Table 1.

$r_{p,l}$	$l \equiv 1$	5	9	13	17	21 (mod 24)
$p \equiv 1$	6	1	2	3	2	1
5	1	1	1	1	1	1
9	2	1	2	1	2	1
13	3	1	1	3	1	1
17	2	1	2	1	2	1
21	1	1	1	1	1	1

Table 1. $r_{p,l}$ for p, l taken modulo 24

The structure of $\Gamma_{p,l}^{ab}$ also seems to depend only on the number $t_{p,l}$ defined in Section 1. Observe that

$$3 \le t_{p,l} \le \min\left\{\frac{p+1}{2}, \frac{l+1}{2}\right\},$$

if $p, l \equiv 1 \pmod 4$.

Conjecture 3. *Let* $p, l \equiv 1 \pmod 4$*. Then*

$$t_{p,l} \equiv \begin{cases} 3 \pmod{12}, & \text{if } r_{p,l} = 1 \\ 9 \pmod{12}, & \text{if } r_{p,l} = 2 \\ 7 \pmod{12}, & \text{if } r_{p,l} = 3 \\ 1 \pmod{12}, & \text{if } r_{p,l} = 6\,. \end{cases}$$

We have checked Conjecture 3 for all pairs of distinct prime numbers $p, l < 1000$ such that $p, l \equiv 1 \pmod 4$. The following values for $t_{p,l}$ appear in this range:

$$t_{p,l} \in \begin{cases} \{3, 15, 27, 39, 51, 63, 75, 87, 99\}\,, & \text{if } r_{p,l} = 1 \\ \{9, 21, 33, 45, 57, 69, 81, 93, 105, 117, 129, 153\}\,, & \text{if } r_{p,l} = 2 \\ \{7, 19, 31, 43, 55, 67, 79, 91, 103, 115, 127, 151\}\,, & \text{if } r_{p,l} = 3 \\ \{37, 49, 61, 73, 85, 97, 109, 121, 133\}\,, & \text{if } r_{p,l} = 6\,. \end{cases}$$

See Table 2 for the frequencies of the values of $t_{p,l}$, where $p, l \equiv 1 \pmod 4$ are prime numbers such that $p < l < 1000$.

$t_{p,l}$	**3**	**15**	**27**	**39**	**51**	**63**	**75**	
#	1242	449	143	56	34	17	7	
	87	**99**						
	5	2						1955
$t_{p,l}$	**9**	**21**	**33**	**45**	**57**	**69**	**81**	
#	178	158	84	57	40	21	8	
	93	**105**	**117**	**129**	141	**153**		
	9	12	5	2		1		575
$t_{p,l}$	**7**	**19**	**31**	**43**	**55**	**67**	**79**	
#	236	130	79	42	18	8	12	
	91	**103**	**115**	**127**	139	**151**		
	6	1	4	2		1		539
$t_{p,l}$	1	13	25	**37**	**49**	**61**	**73**	
#				26	15	15	16	
	85	**97**	**109**	**121**	**133**			
	7	4	3	2	3			91
								3160

TABLE 2. $t_{p,l}$ and its frequency, $p < l < 1000$

Combining Conjecture 3 with Conjecture 1, we get another conjecture:

Conjecture 4. *Let* $p, l \equiv 1 \pmod 4$. *Then*

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_4^3\,, & \text{if } t_{p,l} \equiv 3 \pmod{12} \\ \mathbb{Z}_2^3 \times \mathbb{Z}_8^2\,, & \text{if } t_{p,l} \equiv 9 \pmod{12} \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3\,, & \text{if } t_{p,l} \equiv 7 \pmod{12} \\ \mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2\,, & \text{if } t_{p,l} \equiv 1 \pmod{12}\,. \end{cases}$$

3. THE CASE $p, l \equiv 3 \pmod 4$

If $p, l \equiv 3 \pmod 4$, we have a conjecture similar to Conjecture 2.

Conjecture 5. *Let $p, l \equiv 3 \pmod 4$.*
If $p \pmod 8 = l \pmod 8$, then

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2\,, & \text{if } p, l \equiv 1 \pmod 3 \quad (=: \text{case (B1)}) \\ \mathbb{Z}_2 \times \mathbb{Z}_8^2\,, & \text{else} \quad (=: \text{case (B2)})\,. \end{cases}$$

If $p \pmod 8 \neq l \pmod 8$, then

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^2\,, & \text{if } p, l \equiv 1 \pmod 3 \quad (=: \text{case (B3)}) \\ \mathbb{Z}_2 \times \mathbb{Z}_4^2\,, & \text{else} \quad (=: \text{case (B4)})\,. \end{cases}$$

The four cases (B1)–(B4) defined in the conjecture above can also be expressed taking p and l modulo 24, see Table 3.

	$l \equiv 3$	7	11	15	19	23 $\pmod{24}$
$p \equiv 3$	(B2)	(B4)	(B2)	(B4)	(B2)	(B4)
7	(B4)	(B1)	(B4)	(B2)	(B3)	(B2)
11	(B2)	(B4)	(B2)	(B4)	(B2)	(B4)
15	(B4)	(B2)	(B4)	(B2)	(B4)	(B2)
19	(B2)	(B3)	(B2)	(B4)	(B1)	(B4)
23	(B4)	(B2)	(B4)	(B2)	(B4)	(B2)

TABLE 3. Cases (B1)–(B4) for p, l taken modulo 24

The connection to $t_{p,l}$ is not as nice as in Section 2. We get the following values for $t_{p,l}$, if $p, l \equiv 3 \pmod 4$ are distinct prime numbers less than 1000.

$$t_{p,l} \in \begin{cases} (\{4, 6, \ldots, 104\} \cup \{110, 114, 122, 124, 132\}) \setminus \{84, 88\}\,, & \text{in case (B1)} \\ \{0, 2, \ldots, 78\} \cup \{84, 100, 110\}\,, & \text{in case (B2)} \\ \{0\}\,, & \text{in case (B3)} \\ \{0\}\,, & \text{in case (B4)}\,. \end{cases}$$

In general, i.e. without the restriction $p, l < 1000$, it is easy to see that $t_{p,l}$ is always even. Moreover, it follows from [4, Section 5] that $t_{p,l} = 0$ in the cases (B3), (B4), and $t_{p,l} > 0$ in case (B1). The computations of $t_{p,l}$ combined with Conjecture 5 lead to the following conjecture:

Conjecture 6. *Let $p, l \equiv 3 \pmod 4$.*

(1) *If $t_{p,l} = 0$, then $\Gamma_{p,l}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$ or $\Gamma_{p,l}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^2$ or $\Gamma_{p,l}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^2$.*
(2) *If $t_{p,l} = 2$, then $\Gamma_{p,l}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$.*
(3) *If $t_{p,l} \geq 4$, then $\Gamma_{p,l}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$ or $\Gamma_{p,l}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$.*
(4) *If $\Gamma_{p,l}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^2$ or $\Gamma_{p,l}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^2$, then $t_{p,l} = 0$.*
(5) *If $\Gamma_{p,l}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$, then $t_{p,l} \geq 4$.*

4. The case $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$

The remaining case is $p \pmod 4 \neq l \pmod 4$. Since $\Gamma_{p,l} \cong \Gamma_{l,p}$, we can restrict to $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$.

Conjecture 7. *Let $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$.*
If $l \equiv 1 \pmod 8$, then

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2\,, & \textit{if } p,l \equiv 1 \pmod 3 \quad (=: \textit{case (C1)}) \\ \mathbb{Z}_2 \times \mathbb{Z}_8^2\,, & \textit{else} \quad (=: \textit{case (C2)})\,. \end{cases}$$

If $l \equiv 5 \pmod 8$, then

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^2\,, & \textit{if } p,l \equiv 1 \pmod 3 \quad (=: \textit{case (C3)}) \\ \mathbb{Z}_2 \times \mathbb{Z}_4^2\,, & \textit{else} \quad (=: \textit{case (C4)})\,. \end{cases}$$

Observe that the four conjectured possibilities for $\Gamma_{p,l}^{ab}$ are exactly the same as in Conjecture 5.

See Table 4 for the cases (C1)–(C4) expressed by p and l taken modulo 24.

	$l \equiv 1$	5	9	13	17	21 $\pmod{24}$
$p \equiv 3$	(C2)	(C4)	(C2)	(C4)	(C2)	(C4)
7	(C1)	(C4)	(C2)	(C3)	(C2)	(C4)
11	(C2)	(C4)	(C2)	(C4)	(C2)	(C4)
15	(C2)	(C4)	(C2)	(C4)	(C2)	(C4)
19	(C1)	(C4)	(C2)	(C3)	(C2)	(C4)
23	(C2)	(C4)	(C2)	(C4)	(C2)	(C4)

Table 4. Cases (C1)–(C4) for p, l taken modulo 24

The behaviour of $t_{p,l}$ seems to be very similar as in Section 3. We get the following values for $t_{p,l}$, if $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$ are prime numbers less than 1000.

$$t_{p,l} \in \begin{cases} (\{4,6,\dots,48\} \cup \{58\}) \setminus \{40\}\,, & \text{in case (C1)} \\ \{0,2,\dots,54\} \cup \{60\}\,, & \text{in case (C2)} \\ \{0\}\,, & \text{in case (C3)} \\ \{0\}\,, & \text{in case (C4)}\,. \end{cases}$$

Conjecture 8. *Conjecture 6 also holds if $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$.*

5. More conjectures

In this section, we give conjectures for the abelianization of the commutator subgroup $[\Gamma_{p,l}, \Gamma_{p,l}]$ of $\Gamma_{p,l}$ and for a certain subgroup $\Lambda_{p,l}$ of $\Gamma_{p,l}$ of index 4 defined below.

Conjecture 9. *Let* $p, l \equiv 1 \pmod 4$.
If $p, l \equiv 1 \pmod 8$, *then*

$$[\Gamma_{p,l}, \Gamma_{p,l}]^{ab} \cong \begin{cases} \mathbb{Z}_2^2 \times \mathbb{Z}_{16}^2 \times \mathbb{Z}_{64}\,, & \textit{if } p, l \equiv 1 \pmod 3 \\ \mathbb{Z}_3 \times \mathbb{Z}_{16}^2 \times \mathbb{Z}_{64}\,, & \textit{else}\,. \end{cases}$$

If $p \equiv 5 \pmod 8$ *or* $l \equiv 5 \pmod 8$, *then*

$$[\Gamma_{p,l}, \Gamma_{p,l}]^{ab} \cong \begin{cases} \mathbb{Z}_2^2 \times \mathbb{Z}_{16}^3\,, & \textit{if } p, l \equiv 1 \pmod 3 \\ \mathbb{Z}_3 \times \mathbb{Z}_{16}^3\,, & \textit{else}\,. \end{cases}$$

Conjecture 10. *Let* $p, l \equiv 3 \pmod 4$.
If $p \pmod 8 = l \pmod 8$, *then*

$$[\Gamma_{p,l}, \Gamma_{p,l}]^{ab} \cong \begin{cases} \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}\,, & \textit{if } p, l \equiv 1 \pmod 3 \\ \mathbb{Z}_8^2 \times \mathbb{Z}_{64}\,, & \textit{if } p = 3 \textit{ or } l = 3 \\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}\,, & \textit{else}\,. \end{cases}$$

If $p \pmod 8 \neq l \pmod 8$, *then*

$$[\Gamma_{p,l}, \Gamma_{p,l}]^{ab} \cong \begin{cases} \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}\,, & \textit{if } p, l \equiv 1 \pmod 3 \\ \mathbb{Z}_8^2 \times \mathbb{Z}_{16} & \textit{if } p = 3 \textit{ or } l = 3 \\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}\,, & \textit{else}\,. \end{cases}$$

The groups appearing in Conjecture 11 are again the same as in Conjecture 10:

Conjecture 11. *Let* $p \equiv 3 \pmod 4$ *and* $l \equiv 1 \pmod 4$.
If $l \equiv 1 \pmod 8$, *then*

$$[\Gamma_{p,l}, \Gamma_{p,l}]^{ab} \cong \begin{cases} \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}\,, & \textit{if } p, l \equiv 1 \pmod 3 \\ \mathbb{Z}_8^2 \times \mathbb{Z}_{64}\,, & \textit{if } p = 3 \\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}\,, & \textit{else}\,. \end{cases}$$

If $l \equiv 5 \pmod 8$, *then*

$$[\Gamma_{p,l}, \Gamma_{p,l}]^{ab} \cong \begin{cases} \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}\,, & \textit{if } p, l \equiv 1 \pmod 3 \\ \mathbb{Z}_8^2 \times \mathbb{Z}_{16}\,, & \textit{if } p = 3 \\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}\,, & \textit{else}\,. \end{cases}$$

Let $\Lambda_{p,l}$ be the following subgroup of $\Gamma_{p,l}$.

$$\Lambda_{p,l} := \psi_{p,l}(\{x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z});\ x_0 \text{ odd};\ |x|^2 = p^s l^t, s, t \in 2\mathbb{N}_0\})\,.$$

Observe that $\Lambda_{p,l}$ is the kernel of the surjective homomorphism $\Gamma_{p,l} \to \mathbb{Z}_2 \times \mathbb{Z}_2$ determined by

$$\psi_{p,l}(x) \mapsto \begin{cases} (1 + 2\mathbb{Z},\ 0 + 2\mathbb{Z})\,, & \text{if } |x|^2 = p \\ (0 + 2\mathbb{Z},\ 1 + 2\mathbb{Z})\,, & \text{if } |x|^2 = l\,, \end{cases}$$

in particular $\Lambda_{p,l}$ is a normal subgroup of $\Gamma_{p,l}$ of index 4. It seems that the abelianization of $\Lambda_{p,l}$ does not depend on p and l, if $p, l > 3$.

Conjecture 12. *Let p, l be any pair of distinct odd prime numbers. Then*

$$\Lambda_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_8^2\,, & \text{if } p = 3 \text{ or } l = 3 \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2\,, & \text{else}\,. \end{cases}$$

Since the conjectured abelianizations of the groups $\Gamma_{p,l}$, $[\Gamma_{p,l}, \Gamma_{p,l}]$ and $\Lambda_{p,l}$ are never 2-generated, we also get the following conjecture:

Conjecture 13. *Let p, l be any pair of distinct odd prime numbers. Then the groups $\Gamma_{p,l}$, $[\Gamma_{p,l}, \Gamma_{p,l}]$ and $\Lambda_{p,l}$ are not 2-generated.*

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