AN INCOHERENT SIMPLE GROUP

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Abstract. We give an example of a finitely presented simple group containing a finitely generated subgroup which is not finitely presented.

A group is called *coherent*, if every finitely generated subgroup is finitely presented. The class of coherent groups for example includes free groups, surface groups, or 3-manifold groups ([10]). On the other hand, the group $F_2 \times F_2$ is incoherent ([11]), where F_k denotes the free group of rank k. This also follows from the subsequent result of Grunewald:

Proposition 1. *(Grunewald* [5, Proposition B]*) Let F be a free group of rank $k \geq 2$, generated by $\{s_1, \dots, s_k\}$. Let $r_1, \dots, r_m$ be words over $\{s_1, \dots, s_k\}^{\pm 1}$ and R their normal closure $\langle\langle r_1, \dots, r_m \rangle\rangle_F$ in F. Let H be the group with presentation $\langle s_1, \dots, s_k \mid r_1, \dots, r_m \rangle$ and ϕ the canonical epimorphism $\phi : F \to H \cong F/R$. Let $\overline{F}$ be a free group of rank k generated by $\{t_1, \dots, t_k\}$ and ψ the isomorphism $F \to \overline{F}$, mapping s_i to t_i, $i = 1, \dots, k$. Let $\overline{H} = \langle t_1, \dots, t_k \mid \psi(r_1), \dots, \psi(r_m) \rangle$, and $\tilde{\psi} : H \to \overline{H}$ the isomorphism induced by ψ. Finally, let $\overline{\phi}$ be the canonical epimorphism $\overline{F} \to \overline{H} \cong \overline{F}/\psi(R)$ (see the commutative diagram below for a summary of this notation). Suppose that H is infinite and $R \neq \{1\}$. Then the group $\{(s,t) \in F \times \overline{F} : \tilde{\psi}(\phi(s)) = \overline{\phi}(t)\}$ is a subgroup of $F \times \overline{F}$ generated by the $k + m$ elements $(s_1, t_1), \dots, (s_k, t_k), (r_1, 1), \dots, (r_m, 1)$, but it is not finitely presented.*

$$\begin{array}{ccccc}
R = \langle\langle r_1, \dots, r_m \rangle\rangle_F & \hookrightarrow & F = \langle s_1, \dots, s_k \rangle & \overset{\phi}{\twoheadrightarrow} & H \cong F/R \\
\downarrow {\scriptstyle \psi|_R}\ \cong & & \downarrow {\scriptstyle \psi}\ \cong & & \downarrow {\scriptstyle \tilde{\psi}}\ \cong \\
\psi(R) = \langle\langle \psi(r_1), \dots, \psi(r_m) \rangle\rangle_{\overline{F}} & \hookrightarrow & \overline{F} = \langle t_1, \dots, t_k \rangle & \overset{\overline{\phi}}{\twoheadrightarrow} & \overline{H} \cong \overline{F}/\psi(R)
\end{array}$$

Diagram 1. The setup for Proposition 1

Our strategy will be to construct a finitely presented simple group Λ containing a subgroup isomorphic to $F_2 \times F_2$. Proposition 1 then allows us to construct an explicit subgroup of Λ generated by three elements, which is not finitely presented. We first define a group Γ which will contain Λ as a normal subgroup of index 4.

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Let Γ be the group with finite presentation $\langle a_1, \dots, a_6, b_1, \dots, b_5 \mid R_\Gamma \rangle$, where

$$R_\Gamma := \left\{ \begin{array}{lllll} a_1 b_1 a_2^{-1} b_2^{-1}, & a_1 b_2 a_1^{-1} b_1^{-1}, & a_1 b_3 a_2^{-1} b_3^{-1}, & a_1 b_4 a_1^{-1} b_4^{-1}, & a_1 b_5 a_2^{-1} b_5, \\ a_1 b_5^{-1} a_4 b_5^{-1}, & a_1 b_3^{-1} a_2^{-1} b_2, & a_1 b_1^{-1} a_2^{-1} b_3, & a_2 b_2 a_2^{-1} b_1^{-1}, & a_2 b_4 a_2^{-1} b_4^{-1}, \\ a_2 b_5 a_5^{-1} b_5, & a_3 b_1 a_4^{-1} b_2^{-1}, & a_3 b_2 a_3^{-1} b_1^{-1}, & a_3 b_3 a_4^{-1} b_3^{-1}, & a_3 b_4 a_4 b_4, \\ a_3 b_5 a_4 b_4^{-1}, & a_3 b_5^{-1} a_6^{-1} b_5^{-1}, & a_3 b_4^{-1} a_4 b_5, & a_3 b_3^{-1} a_4^{-1} b_2, & a_3 b_1^{-1} a_4^{-1} b_3, \\ a_4 b_2 a_4^{-1} b_1^{-1}, & a_5 b_1 a_5^{-1} b_1^{-1}, & a_5 b_2 a_5 b_3^{-1}, & a_5 b_3 a_6^{-1} b_5, & a_5 b_4 a_5^{-1} b_4^{-1}, \\ a_5 b_5 a_6^{-1} b_2^{-1}, & a_5 b_2^{-1} a_6 b_3, & a_6 b_1 a_6^{-1} b_3, & a_6 b_2 a_6^{-1} b_4^{-1}, & a_6 b_4 a_6^{-1} b_1 \end{array} \right\}.$$

We have found Γ using programs written in GAP([4]). It is constructed such that simultaneously (compare to the proof of Theorem 5)

- $\Gamma < \mathrm{Aut}(\mathcal{T}_{12}) \times \mathrm{Aut}(\mathcal{T}_{10})$, where $\mathrm{Aut}(\mathcal{T}_k)$ denotes the group of automorphisms of the k-regular tree $\mathcal{T}_k$,
- Γ (as well as any finite index subgroup of Γ) does not have any non-trivial normal subgroup of infinite index by a theorem of Burger-Mozes ([3]),
- the subgroup $\langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \rangle_\Gamma$ of Γ is not residually finite by a theorem of Wise ([12, Main Theorem II.5.5]), more precisely the element $a_2 a_1^{-1} a_3 a_4^{-1}$ is contained in each finite index subgroup of $\langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \rangle_\Gamma$ and therefore in each finite index subgroup of Γ,
- the normal closure $\langle\langle a_2 a_1^{-1} a_3 a_4^{-1} \rangle\rangle_\Gamma$ has finite index in Γ,
- the subgroup $\langle a_5, a_1 a_2^{-1}, b_1, b_4 \rangle_\Gamma$ is isomorphic to $F_2 \times F_2$.

The latter statement will follow from Lemma 4 below, using a well-known normal form for elements in Γ:

Lemma 2. *(Bridson-Wise* [2, Normal Form Lemma 4.3]*) Any element $\gamma \in \Gamma$ can be written as $\gamma = \sigma_a \sigma_b = \sigma'_b \sigma'_a$, where σ_a, σ'_a are freely reduced words in the subgroup $\langle a_1, \dots, a_6 \rangle_\Gamma$ and σ_b, σ'_b are freely reduced words in $\langle b_1, \dots, b_5 \rangle_\Gamma$. The words $\sigma_a, \sigma'_a, \sigma_b, \sigma'_b$ are uniquely determined by γ. Moreover, $|\sigma_a| = |\sigma'_a|$ and $|\sigma_b| = |\sigma'_b|$, where $|\cdot|$ is the word length with respect to the standard generators $\{a_1, \dots, a_6, b_1, \dots, b_5\}^{\pm 1}$ of Γ.*

Note that Lemma 2 was proved in [2] for a certain class of fundamental groups of square complexes covered by a product of trees. The following two lemmas are a direct consequence of the uniqueness statement in Lemma 2.

Lemma 3. *The subgroup $\langle a_1, \dots, a_6 \rangle_\Gamma$ is a free group of rank 6, and $\langle b_1, \dots, b_5 \rangle_\Gamma$ is a free group of rank 5.*

Lemma 4. *Let $a, \tilde{a} \in \langle a_1, \dots, a_6 \rangle_\Gamma$ and $b, \tilde{b} \in \langle b_1, \dots, b_5 \rangle_\Gamma$, such that $ab = ba$, $a\tilde{b} = \tilde{b}a$, $\tilde{a}b = b\tilde{a}$ and $\tilde{a}\tilde{b} = \tilde{b}\tilde{a}$. Then the map $\langle a, \tilde{a}, b, \tilde{b} \rangle_\Gamma \to \langle a, \tilde{a} \rangle_\Gamma \times \langle b, \tilde{b} \rangle_\Gamma$, given by $a \mapsto (a, 1)$, $\tilde{a} \mapsto (\tilde{a}, 1)$, $b \mapsto (1, b)$, $\tilde{b} \mapsto (1, \tilde{b})$ is an isomorphism of groups. In particular, if moreover $\langle a, \tilde{a} \rangle_\Gamma \cong F_2$ and $\langle b, \tilde{b} \rangle_\Gamma \cong F_2$, then $\langle a, \tilde{a}, b, \tilde{b} \rangle_\Gamma \cong F_2 \times F_2$.*

We define our main group Λ to be the kernel of the surjective homomorphism of groups

$$\begin{aligned}\varphi : \Gamma &\to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\\ a_1, \dots, a_6 &\mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}),\\ b_1, \dots, b_5 &\mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}).\end{aligned}$$

Theorem 5. *The finitely presented group Λ is simple and incoherent. More precisely, the subgroup $\langle a_5^2 b_1^2, a_1 a_2^{-1} b_4^2, a_5^2 \rangle_\Gamma < \Lambda$ is not finitely presented.*

Proof. The simplicity of Λ follows similarly as in [8, Theorem 3.5], but we recall the main steps in the proof. First note that Γ is the fundamental group of a finite square complex X having a single vertex called x, having $6 + 5$ oriented loops (identified with $a_1, \dots, a_6$, $b_1, \dots, b_5$) and $6 \cdot 5$ squares (identified with the relators in R_Γ). Those 30 squares are carefully chosen such that several conditions simultaneously hold. For example, the link of x in X is a complete bipartite graph on $12 + 10$ vertices which correspond to $\{a_1, \dots, a_6\}^{\pm 1}$ and $\{b_1, \dots, b_5\}^{\pm 1}$. As a consequence, the universal covering space $\tilde{X}$ is a product of two regular trees $\mathcal{T}_{12} \times \mathcal{T}_{10}$, and Γ is a subgroup of $\mathrm{Aut}(\mathcal{T}_{12}) \times \mathrm{Aut}(\mathcal{T}_{10})$. The local actions on $\mathcal{T}_{12}$ of the projection of Γ to the first factor $\mathrm{Aut}(\mathcal{T}_{12})$ are described by finite permutation groups $P_h^{(k)}(\Gamma) < S_{12 \cdot 11^{k-1}}$, where $k \in \mathbb{N}$ (see [3, Chapter 1] or [8]). We compute for $k = 1$

$$\begin{aligned}P_h^{(1)}(\Gamma) = \langle &(9,10)(11,12), (1,2)(3,4)(5,6,8), (1,2)(3,4)(5,8,7)(9,10)(11,12),\\ &(3,9)(4,10), (1,9,3,6,5,2)(4,12,11,8,7,10) \rangle = A_{12}\end{aligned}$$

and similarly (taking the projection to the second factor of $\mathrm{Aut}(\mathcal{T}_{12}) \times \mathrm{Aut}(\mathcal{T}_{10})$, thus getting finite permutation groups $P_v^{(k)}(\Gamma) < S_{10 \cdot 9^{k-1}}$)

$$\begin{aligned}P_v^{(1)}(\Gamma) = \langle &(1,2)(5,6)(8,10,9), (1,2,3)(5,6)(9,10), (1,2)(4,5,6,7)(8,10,9),\\ &(1,2,3)(4,5,6,7)(9,10), (2,5,6,3)(8,9), (1,7,9,6,5,8)(2,3,10,4) \rangle = A_{10}.\end{aligned}$$

By the Normal Subgroup Theorem of Burger-Mozes ([3], see also [8]), using the simplicity and high transitivity of A_{12} and A_{10}, the group Γ has no non-trivial normal subgroups of infinite index. This theorem can also be applied to any finite index subgroup of Γ, in particular to $\Lambda < \Gamma$ which is a subgroup of index 4 by definition.

Next we have to study the *finite* index subgroups of Λ. We start with the subgroup $\langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \rangle_\Gamma$ of Γ. It has a presentation

$$\begin{aligned}\langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \mid\ &a_1 b_1 a_2^{-1} b_2^{-1},\ a_1 b_2 a_1^{-1} b_1^{-1},\ a_1 b_3 a_2^{-1} b_3^{-1},\ a_1 b_3^{-1} a_2^{-1} b_2,\\ &a_1 b_1^{-1} a_2^{-1} b_3,\ a_2 b_2 a_2^{-1} b_1^{-1},\ a_3 b_1 a_4^{-1} b_2^{-1},\ a_3 b_2 a_3^{-1} b_1^{-1},\\ &a_3 b_3 a_4^{-1} b_3^{-1},\ a_3 b_3^{-1} a_4^{-1} b_2,\ a_3 b_1^{-1} a_4^{-1} b_3,\ a_4 b_2 a_4^{-1} b_1^{-1} \rangle,\end{aligned}$$

since it is the fundamental group $\pi_1(W, x)$ of a finite square complex W which is embedded in X by construction. This implies that $\pi_1(W, x) < \Gamma = \pi_1(X, x)$. Observe that the 12 relators in the presentation of $\pi_1(W, x)$ also appear in the presentation of Γ, and that the link of the single vertex (again called x) in W is a complete bipartite graph on $8 + 6$ vertices corresponding to $\{a_1, a_2, a_3, a_4\}^{\pm 1}$ and $\{b_1, b_2, b_3\}^{\pm 1}$. This group $\pi_1(W, x)$ is not residually finite and was introduced exactly for this purpose by Wise in [12] where it is called $\pi_1(D)$. He showed for example that the element $a_2a_1^{-1}a_3a_4^{-1}$ is contained in each finite index subgroup of $\pi_1(W,x)$. Consequently, this element is also contained in each finite index subgroup of Γ. Since $\langle\langle a_2a_1^{-1}a_3a_4^{-1}\rangle\rangle_\Gamma$ has index 4 in Γ, it follows (see [8]) that

$$\Lambda = \langle\langle a_2a_1^{-1}a_3a_4^{-1}\rangle\rangle_\Gamma = \bigcap\{N \lhd \Gamma : N \text{ has finite index in } \Gamma\},$$

but the latter group is easily seen to have no proper subgroups of finite index, hence Λ is simple.

We show now that Λ is incoherent. First observe that $a_1a_2^{-1}$ commutes with b_1 in $\pi_1(W,x)$ (and therefore in Γ), since

$$a_1a_2^{-1}b_1 = a_1b_2a_2^{-1} = b_1a_1a_2^{-1},$$

using the square relators $a_2b_2a_2^{-1}b_1^{-1}$ and $a_1b_2a_1^{-1}b_1^{-1}$ from the presentation of $\pi_1(W,x)$. Moreover, we have forced X to contain certain tori, namely $a_1b_4a_1^{-1}b_4^{-1}$, $a_2b_4a_2^{-1}b_4^{-1}$, $a_5b_1a_5^{-1}b_1^{-1}$ and $a_5b_4a_5^{-1}b_4^{-1}$. This implies that

$$a_5b_1 = b_1a_5,\ a_5b_4 = b_4a_5 \text{ and } a_1a_2^{-1}b_4 = b_4a_1a_2^{-1}$$

holds in Γ. By Lemma 4 and Lemma 3,

$$\langle a_5, a_1a_2^{-1}, b_1, b_4\rangle_\Gamma \cong \langle a_5, a_1a_2^{-1}\rangle_\Gamma \times \langle b_1, b_4\rangle_\Gamma \cong F_2 \times F_2,$$

but this group is not contained in Λ (recall the definition of Λ as kernel of φ). Therefore, we take the subgroup

$$\langle a_5^2, a_1a_2^{-1}, b_1^2, b_4^2\rangle_\Gamma \cong \langle a_5^2, a_1a_2^{-1}\rangle_\Gamma \times \langle b_1^2, b_4^2\rangle_\Gamma \cong F_2 \times F_2,$$

which is obviously a subgroup of Λ. To see that $\langle a_5^2b_1^2, a_1a_2^{-1}b_4^2, a_5^2\rangle_\Gamma$ is not finitely presented, we apply Proposition 1 to the following setting: $k=2$, $s_1 = a_5^2$, $s_2 = a_1a_2^{-1}$, $t_1 = b_1^2$, $t_2 = b_4^2$, and $m=1$, $r_1 = s_1$, that is we take $F = \langle a_5^2, a_1a_2^{-1}\rangle_\Gamma \cong F_2$ and $\overline{F} = \langle b_1^2, b_4^2\rangle_\Gamma \cong F_2$. It remains to check that H is infinite and $R \neq \{1\}$, but this is clear since $H = \langle s_1, s_2 \mid s_1\rangle \cong \langle a_1a_2^{-1}\rangle_\Gamma \cong \mathbb{Z}$, and $R = \langle\langle a_5^2\rangle\rangle_F$. □

Remark 6. Note that the group Λ can be decomposed as amalgamated products $F_9 *_{F_{97}} F_9$ and $F_{11} *_{F_{101}} F_{11}$ (see [7, Proposition 1.4]), in particular Λ is torsion-free.

Remark 7. It is well-known that the word problem is solvable for any finitely presented simple group. In fact, by a theorem of Boone-Higman ([1]), a finitely generated group has solvable word problem if and only if it can be embedded in a simple subgroup of a finitely presented group. However, the *generalized* word problem is not solvable for the simple group Λ, since it contains $F_2 \times F_2$ (using a result of Mihaĭlova [6]). Recall that the generalized word problem is solvable for a group G if it is decidable for any element $g \in G$ and any finitely generated subgroup $H < G$ whether or not g lies in H. It is also known that Λ has solvable conjugacy problem (being bi-automatic).

Remark 8. We mention two other ways to construct finitely presented incoherent simple groups: One is to directly embed $F_2 \times F_2$ into a virtually simple group by [3, Theorem 6.5]. The second one is a finitely presented simple group containing $\mathrm{GL}_4(\mathbb{Z})$ constructed by E. Scott ([9]). It is known ([5]) that $\mathrm{SL}_4(\mathbb{Z}) < \mathrm{GL}_4(\mathbb{Z})$ is incoherent. In fact, if $A, B \in \mathrm{SL}_2(\mathbb{Z})$ generate a free group of rank 2, and E denotes the identity matrix in $\mathrm{SL}_2(\mathbb{Z})$, then

$$\left\langle \begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & E \end{pmatrix}, \begin{pmatrix} E & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} E & 0 \\ 0 & B \end{pmatrix} \right\rangle_{\mathrm{SL}_4(\mathbb{Z})} \cong F_2 \times F_2.$$

Explicitly, one can take

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

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