PERIODS IN QUATERNION SQUARE COMPLEX GROUPS

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Let (p,l) be any pair of distinct odd prime numbers. Let $x = x_0+x_1i+x_2j+x_3k \in \mathbb{H}(\mathbb{Z}) \setminus \{0\}$. Define

$$\psi_{p,l} : \mathbb{H}(\mathbb{Z}) \setminus \{0\} \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$

by

$$\psi_{p,l}(x) = \left(\begin{pmatrix} x_0 + x_1c_p + x_3d_p & -x_1d_p + x_2 + x_3c_p \\ -x_1d_p - x_2 + x_3c_p & x_0 - x_1c_p - x_3d_p \end{pmatrix}, \begin{pmatrix} x_0 + x_1c_l + x_3d_l & -x_1d_l + x_2 + x_3c_l \\ -x_1d_l - x_2 + x_3c_l & x_0 - x_1c_l - x_3d_l \end{pmatrix} \right),$$

where $c_p, d_p \in \mathbb{Q}_p$ and $c_l, d_l \in \mathbb{Q}_l$ are elements such that $c_p^2 + d_p^2 + 1 = 0$ and $c_l^2+d_l^2+1 = 0$. Note that $\psi_{p,l}(xy) = \psi_{p,l}(x)\psi_{p,l}(y)$, $\psi_{p,l}(\lambda x) = \psi_{p,l}(x)$ if $\lambda \in \mathbb{Z}\setminus\{0\}$, and $(\psi_{p,l}(x))^{-1} = \psi_{p,l}(\overline{x})$, where $\overline{x} = x_0 - x_1i - x_2j - x_3k$ is the conjugate of x. Define the norm $|x|^2 = x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2$.

Let

$$\begin{aligned}\tilde{\Gamma}_{p,l} = \{x \in \mathbb{H}(\mathbb{Z})\,; \quad & |x|^2 = p^r l^s, r, s \geq 0\,; \\ & x_0 \text{ odd}, x_1, x_2, x_3 \text{ even, if } |x|^2 \equiv 1 \pmod 4\,; \\ & x_1 \text{ even}, x_0, x_2, x_3 \text{ odd, if } |x|^2 \equiv 3 \pmod 4\}.\end{aligned}$$

Define subsets $\tilde{\Gamma}_p := \{x \in \tilde{\Gamma}_{p,l} : |x|^2 = p\}$ and $\tilde{\Gamma}_l := \{x \in \tilde{\Gamma}_{p,l} : |x|^2 = l\}$. Note that $|\tilde{\Gamma}_p| = 2(p+1)$ and $|\tilde{\Gamma}_l| = 2(l+1)$. Define $A_p = \{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \psi_{p,l}(\tilde{\Gamma}_p)$, $B_l = \{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \psi_{p,l}(\tilde{\Gamma}_l)$ and the group $\Gamma_{p,l} = \psi_{p,l}(\tilde{\Gamma}_{p,l})$ which is generated by $\{a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}}\}$.

We associate to $\Gamma_{p,l}$ a set of squares ("tiles")

$$T(\Gamma_{p,l}) = \{aba'b' : a, a' \in A_p;\ b, b' \in B_l;\ aba'b' = 1 \in \Gamma_{p,l}\}.$$

We want to define a map $\mathrm{per} : A_p \times B_l \to \mathbb{N}_0$. Fix $(a,b) \in A_p \times B_l$. Then there is a unique pair $(a', b') \in A_p \times B_l$ such that $aba'b' = 1 \in \Gamma_{p,l}$, in particular

$$|T(\Gamma_{p,l})| = |A_p| \cdot |B_l| = (p+1)(l+1).$$

If $a' = a$ or $b' = b$, define $\mathrm{per}(a,b) = 0$.

If $a' \neq a$ and $b' \neq b$, then there is a unique tiling of the plane $f_{(a,b)} : \mathbb{Z}^2 \to T(\Gamma_{p,l})$, such that $f_{(a,b)}(i,i) = aba'b'$ for all $i \in \mathbb{Z}$. In this case define

$$\mathrm{per}(a,b) = \min\{j \in \mathbb{N} : f_{(a,b)}(i, i+j) = aba'b' \text{ for all } i \in \mathbb{Z}\}$$

and its multiplicity (divided by 4)

$$\mathrm{mult}_{p,l}(\mathrm{per}(a,b)) = \frac{1}{4}|\{(\hat{a},\hat{b}) \in A_p \times B_l : \mathrm{per}(\hat{a},\hat{b}) = \mathrm{per}(a,b)\}|.$$

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We take the factor $\frac{1}{4}$, since

$$\mathrm{per}(a,b) = \mathrm{per}(a',b') = \mathrm{per}(a^{-1},b'^{-1}) = \mathrm{per}(a'^{-1},b^{-1}),$$

if $aba'b' = 1$.

In the following table, we list two rows for each pair (p,l) such that $2 < p < l < 100$: in the first row are the possible periods of pairs $(a,b) \in A_p \times B_l$, i.e. the elements of the set

$$\{\mathrm{per}(a,b) : (a,b) \in A_p \times B_l\}$$

and in the second row are the corresponding multiplicities $\mathrm{mult}_{p,l}(\mathrm{per}(a,b))$. Observe that the sum of the numbers in the second row is always $(p+1)(l+1)/4$.

There are 276 pairs (p,l) in the list.

(p,l)															
(3,5)	0	4													
	2	4													
(3,7)	0	4													
	2	6													
(3,11)	0	1	2												
	6	2	4												
(3,13)	0	4	6												
	4	4	6												
(3,17)	0	1	3	6											
	4	2	6	6											
(3,19)	0	1	2	4											
	2	2	4	12											
(3,23)	4	6	8												
	6	6	12												
(3,29)	0	6	8	10											
	6	6	8	10											
(3,31)	0	6	8												
	14	6	12												
(3,37)	0	6	8	10											
	8	12	8	10											
(3,41)	0	1	5	8											
	12	2	20	8											
(3,43)	0	1	2	3	4	6									
	14	2	4	6	6	12									
(3,47)	0	4	6	12											
	12	6	12	18											
(3,53)	0	6	8	14											
	14	18	8	14											
(3,59)	0	1	2	4	6										
	18	2	4	12	24										
(3,61)	0	4	6	8	14										
	16	4	12	16	14										
(3,67)	0	1	2	4	6										
	14	2	4	6	42										
(3,71)	0	6	12	16											
	24	6	18	24											
(3,73)	0	1	5	6	8										
	16	2	30	18	8										
(3,79)	0	8	12	18											
	14	12	36	18											
(3,83)	0	1	2	4	6	8									
	18	2	4	18	18	24									

(3,89)	0	1	5	8	10										
	24	2	20	24	20										
(3,97)	0	1	5	6	10										
	24	2	20	12	40										
(5,7)	0	4	6												
	2	4	6												
(5,11)	0	2	4	6											
	4	4	4	6											
(5,13)	0	1	4												
	6	3	12												
(5,17)	1	3	4												
	3	6	18												
(5,19)	0	4	8	10											
	4	8	8	10											
(5,23)	0	4	6												
	10	20	6												
(5,29)	0	1	2	4											
	12	3	12	18											
(5,31)	0	4	6	10	16										
	4	12	6	10	16										
(5,37)	0	1	3	4	6										
	6	3	6	24	18										
(5,41)	0	1	4	6	8										
	18	3	18	12	12										
(5,43)	0	4	6	8	10	14	20								
	4	4	6	8	10	14	20								
(5,47)	0	4	6	8											
	10	24	30	8											
(5,53)	0	1	2	4	6										
	6	3	12	42	18										
(5,59)	0	2	4	6											
	20	4	12	54											
(5,61)	0	1	3	4	10										
	18	3	6	36	30										
(5,67)	0	4	6	10	12	16	22								
	20	4	18	10	12	16	22								
(5,71)	0	4	6	8	10	12									
	12	12	30	32	10	12									
(5,73)	0	1	3	6	8										
	24	3	6	66	12										
(5,79)	0	4	6	8	10	12	20	26							
	24	4	6	8	20	12	20	26							
(5,83)	0	2	4	6	8	12									
	20	4	16	42	32	12									
(5,89)	0	1	3	4	6	8	12								
	24	3	6	18	42	24	18								
(5,97)	0	1	3	6	7	8									
	36	3	6	48	42	12									
(7,11)	4	6	8												
	6	6	12												
(7,13)	0	2	4												
	4	4	20												
(7,17)	0	3	4	6											
	4	12	8	12											

(7,19)	0	2	4	6											
	12	4	18	6											
(7,23)	2	3	4	6											
	12	6	18	12											
(7,29)	0	4	6	8	10	20									
	8	8	6	8	10	20									
(7,31)	0	1	2	3	4	6									
	12	4	12	6	18	12									
(7,37)	0	2	4	6	8	10									
	8	4	28	18	8	10									
(7,41)	0	3	4	5	6	7									
	8	12	8	30	12	14									
(7,43)	2	4	6	8											
	4	30	18	36											
(7,47)	0	2	4	6											
	12	12	48	24											
(7,53)	0	2	4	6	10	16									
	8	4	4	36	40	16									
(7,59)	0	4	8	12	18	28									
	12	6	24	18	18	42									
(7,61)	0	4	6	8	10	12									
	16	36	6	32	10	24									
(7,67)	0	2	4	6	8	12									
	24	4	48	30	12	18									
(7,71)	0	3	4	6	8	10									
	12	6	54	30	12	30									
(7,73)	0	1	3	4	6	7	9								
	12	4	36	40	24	14	18								
(7,79)	0	1	2	3	4	6									
	36	4	12	6	84	18									
(7,83)	0	4	6	8	18	20									
	24	36	12	48	18	30									
(7,89)	0	3	4	5	6	7	17								
	28	18	16	20	36	28	34								
(7,97)	0	1	3	4	5	6	7								
	28	4	18	72	20	12	42								
(11,13)	0	4	6	8	14										
	2	4	6	16	14										
(11,17)	0	1	2	3	4	6									
	8	2	4	6	16	18									
(11,19)	0	1	2	3	4	6									
	6	6	12	6	18	12									
(11,23)	0	4	6	12											
	12	24	18	18											
(11,29)	0	2	4	6	8										
	10	12	24	36	8										
(11,31)	4	6	8	12	18										
	18	6	36	18	18										
(11,37)	0	4	6	8	20	22									
	12	28	24	8	20	22									
(11,41)	0	1	2	3	4	6	12								
	8	6	4	18	48	18	24								
(11,43)	0	1	2	3	4	6	8	12							
	6	2	4	6	36	36	24	18							
(11,47)	0	4	6	8	12										
	18	42	18	12	54										

(11,53)	0	2	4	6	8	10	14								
	8	4	24	36	56	20	14								
(11,59)	0	1	2	3	4	6									
	12	6	60	6	66	30									
(11,61)	0	4	6	8	10	12	14	22	32						
	14	16	12	24	40	12	14	22	32						
(11,67)	0	1	2	4	5	6	8	10	16						
	24	2	4	54	10	42	24	20	24						
(11,71)	0	4	6	8	12	18									
	24	54	36	48	36	18									
(11,73)	0	1	3	4	5	6	7	8	9	10	16				
	12	2	12	16	10	12	70	16	36	20	16				
(11,79)	0	4	6	8	12	18	20								
	12	60	12	72	18	36	30								
(11,83)	0	1	2	4	6										
	30	2	28	78	114										
(11,89)	0	1	2	3	4	5	6	10	12						
	16	6	4	24	72	20	84	20	24						
(11,97)	0	1	3	4	5	6	7	8	9	10	13	14	18	22	
	16	2	42	16	10	12	14	24	18	20	26	14	36	44	
(13,17)	0	1	3	4	6	8									
	6	3	24	6	12	12									
(13,19)	0	2	4	6	10										
	4	4	28	24	10										
(13,23)	0	2	4	6	8	10									
	4	4	12	6	8	50									
(13,29)	0	1	2	4	6	8	10								
	6	3	12	12	18	24	30								
(13,31)	0	2	4	6	8	10									
	18	8	24	12	40	10									
(13,37)	0	1	2	3	4	6									
	12	7	36	6	54	18									
(13,41)	0	1	3	4	8	12	15								
	12	3	24	12	48	18	30								
(13,43)	0	2	4	6	8	10									
	16	12	44	30	32	20									
(13,47)	0	2	4	6	8	10	14	18	20						
	8	4	36	36	8	10	28	18	20						
(13,53)	0	1	2	3	4	6	8	10							
	12	3	12	6	30	72	24	30							
(13,59)	0	4	6	8	12	22	24	28							
	8	40	6	24	12	44	48	28							
(13,61)	1	2	3	4	6	8									
	7	36	6	96	36	36									
(13,67)	0	2	4	6	8	10	16								
	16	8	68	42	32	40	32								
(13,71)	0	4	6	8	10	12	14	18	20	26	36				
	20	24	6	8	30	24	14	18	20	52	36				
(13,73)	0	1	2	3	4	6	8	10							
	24	7	12	6	78	78	24	30							
(13,79)	0	4	6	8	10	12	14	18							
	16	52	18	64	30	36	28	36							
(13,83)	0	2	4	6	8	10	12	14	16	18	22	26			
	16	4	28	30	48	20	12	14	16	36	44	26			
(13,89)	1	3	4	6	8	9	12	15	22						
	3	18	24	54	60	36	24	30	66						

(13,97)	0	1	2	3	4	6	8	12							
	24	7	12	66	90	48	72	24							
(17,19)	0	1	3	4	8	10	13								
	4	2	24	16	8	10	26								
(17,23)	0	2	3	4	5	6									
	4	8	6	32	10	48									
(17,29)	0	1	2	3	4	6	12								
	12	3	12	6	54	12	36								
(17,31)	0	3	4	5	7	8	16								
	8	30	8	20	14	32	32								
(17,37)	0	1	3	4	6	8	12	14							
	12	3	24	30	12	24	24	42							
(17,41)	0	1	2	3	4	6									
	12	9	36	24	54	54									
(17,43)	0	1	3	4	6	10	12	14							
	8	2	36	8	48	20	48	28							
(17,47)	0	2	3	4	5	6	8	10							
	16	16	6	56	10	36	16	60							
(17,53)	0	1	3	4	6	8	12								
	36	3	24	72	36	48	24								
(17,59)	0	1	2	3	4	6	7	8	10	12					
	24	2	12	36	40	42	14	56	20	24					
(17,61)	0	1	3	4	7	8	10	16							
	12	3	24	60	42	60	30	48							
(17,67)	0	1	3	4	6	7	8	11	13	22	24				
	20	2	36	24	78	14	16	22	26	44	24				
(17,71)	0	3	4	5	6	7	8	10	11	14					
	16	42	40	10	36	14	48	40	22	56					
(17,73)	0	1	2	3	4	6	8	10							
	24	9	24	48	126	18	24	60							
(17,79)	0	3	4	5	6	7	10	14	15	16	17	18			
	16	54	24	20	24	14	20	56	30	32	34	36			
(17,83)	0	1	2	3	4	6	7	8							
	16	2	12	24	128	126	14	56							
(17,89)	0	1	2	3	4	6	8								
	60	9	36	42	102	144	12								
(17,97)	0	1	2	3	4	6	8	10	12						
	12	9	24	78	54	48	120	60	36						
(19,23)	0	4	6	8	12	18	24								
	6	6	6	12	36	18	36								
(19,29)	0	2	4	6	8	10	14	22							
	4	4	12	42	32	20	14	22							
(19,31)	0	2	4	6	8										
	12	4	84	36	24										
(19,37)	0	2	4	6	8	10	14								
	18	8	16	30	40	50	28								
(19,41)	0	1	3	4	5	6	7	8	10	11	13	14			
	8	6	30	16	10	12	14	32	20	22	26	14			
(19,43)	0	1	2	3	4	6	8								
	24	6	28	12	54	72	24								
(19,47)	0	4	6	8	12	16	18	20							
	12	78	18	12	24	48	18	30							
(19,53)	0	4	8	10	12	14	16	18	20	26	30	40			
	16	16	16	20	24	28	16	18	20	26	30	40			

(19,59)	0	1	2	3	4	5	6	8	10	14					
	12	6	36	24	48	10	54	48	20	42					
(19,61)	0	2	4	6	8	10	12	26							
	16	12	84	72	32	20	48	26							
(19,67)	0	1	2	3	4	6	8								
	12	6	64	24	66	144	24								
(19,71)	4	6	8	16	18	20	30								
	66	42	120	24	18	60	30								
(19,73)	0	1	3	4	5	6	8	9	10						
	32	6	48	80	50	84	32	18	20						
(19,79)	0	2	4	6	8	12	16	20							
	48	4	90	72	60	24	72	30							
(19,83)	0	1	2	4	5	6	8	10	12	14					
	6	2	16	102	20	30	60	40	18	126					
(19,89)	0	1	2	3	4	5	6	7	10	12	14	15	29	30	
	24	6	8	30	64	40	36	28	20	48	28	30	58	30	
(19,97)	0	1	3	4	5	6	7	8	9	10	11	12			
	24	6	60	88	20	120	14	56	36	20	22	24			
(23,29)	0	2	4	6	8	10	18								
	8	8	28	48	40	30	18								
(23,31)	0	1	2	4	5	6	8	10							
	12	4	8	42	10	30	36	50							
(23,37)	0	2	4	6	8	10	26	36							
	8	4	32	48	64	10	26	36							
(23,41)	0	2	3	4	5	6	7	8	10	13	14				
	20	8	18	8	10	36	14	16	40	26	56				
(23,43)	0	4	6	8	12	16	24	30							
	12	36	24	48	54	24	36	30							
(23,47)	0	1	2	3	4	6	10								
	24	4	44	6	126	54	30								
(23,53)	0	2	4	6	10	12	14								
	16	20	64	96	50	36	42								
(23,59)	2	4	6	12	16	18	20								
	24	66	144	54	24	18	30								
(23,61)	0	4	8	10	12	14	22	24	26						
	20	24	80	20	12	98	44	48	26						
(23,67)	0	4	6	8	12	16	18	20							
	12	78	36	72	24	48	18	120							
(23,71)	0	1	2	4	6	8									
	24	4	56	120	144	84									
(23,73)	0	3	4	5	7	8	10	11	14	15	18				
	20	30	64	10	70	16	60	22	56	60	36				
(23,79)	0	3	4	5	6	7	8	10	12	14	16				
	24	18	66	30	84	14	12	90	90	28	24				
(23,83)	0	2	4	6	8	10	12	16	20						
	24	12	150	42	12	60	150	24	30						
(23,89)	0	1	3	4	5	6	7	8	9	10	12	14	18	22	
	24	4	42	72	10	36	42	48	18	60	48	56	36	44	
(23,97)	0	1	3	4	5	6	7	10	11	13	14	16	20	24	26
	20	4	18	32	30	48	28	80	44	52	28	64	40	48	52
(29,31)	0	4	6	8	10	12	14	24	26	28					
	2	40	48	16	30	12	14	24	26	28					
(29,37)	0	1	3	4	6	8	10								
	6	3	6	48	36	96	90								

(29,41)	1	2	3	4	6	8	12								
	3	12	48	78	42	72	60								
(29,43)	0	2	4	6	8	12	14	16	22						
	16	4	44	120	24	24	28	48	22						
(29,47)	0	2	4	6	8	10	12	14	16	22					
	10	16	32	42	48	20	48	42	80	22					
(29,53)	0	1	2	3	4	6	8	12							
	36	3	60	12	42	144	72	36							
(29,59)	0	2	4	6	8	10	12	14	22						
	16	32	100	36	24	130	12	56	44						
(29,61)	0	1	2	3	4	6	8	9	10						
	6	3	24	18	144	198	24	18	30						
(29,67)	0	4	6	8	10	12	14	16	18	20	22	30	40	46	52
	20	32	12	48	30	12	28	16	36	20	88	30	40	46	52
(29,71)	0	2	4	6	8	10	14	16	18						
	24	24	124	120	72	80	28	32	36						
(29,73)	1	3	4	5	6	8	9	12	14	15	16	27			
	3	42	36	30	102	24	18	72	42	60	72	54			
(29,79)	0	4	6	8	10	14	18	20	24	56					
	14	72	72	48	130	84	36	40	48	56					
(29,83)	0	2	4	6	8	10	14	16	22	30	36				
	48	24	84	108	32	90	56	48	44	60	36				
(29,89)	0	1	2	3	4	5	6	9	10	12	16	20			
	24	15	36	6	126	30	96	36	30	60	96	120			
(29,97)	0	1	3	4	5	6	8	9	12	15	18	20	32	34	
	36	3	72	24	90	54	72	18	24	60	54	30	96	102	
(31,37)	0	2	4	6	8	10	12	16	18	24					
	12	4	96	12	32	50	24	32	18	24					
(31,41)	0	2	3	4	5	6	8	9	10	12	21				
	8	8	12	56	50	24	16	36	60	24	42				
(31,43)	0	2	4	6	12	16	18	20							
	12	16	60	90	72	24	18	60							
(31,47)	1	2	3	4	5	6	8	10	12	18					
	4	20	6	54	10	96	84	20	36	54					
(31,53)	0	4	6	8	10	12	14	22	24	32	34	36			
	12	28	18	16	10	48	42	66	24	64	68	36			
(31,59)	0	4	6	8	10	12	16	18	20						
	24	60	54	108	30	36	72	36	60						
(31,61)	0	2	4	6	8	10	12	16	18						
	24	28	60	144	24	90	12	96	18						
(31,67)	2	4	6	8	10	12	16	18	32						
	28	72	180	72	60	24	24	36	48						
(31,71)	0	1	2	3	4	6	7	10	14	16					
	24	4	32	24	96	102	14	60	196	24					
(31,73)	0	1	2	3	4	6	7	8	9	10	11	18			
	16	4	24	114	112	60	84	64	36	20	22	36			
(31,79)	0	1	2	3	4	6	7	8	14	18					
	12	12	88	36	174	126	14	132	28	18					
(31,83)	0	4	6	8	12	16	18	20	24	30	40	42	56	60	
	12	42	18	48	36	48	18	60	84	30	60	42	84	90	
(31,89)	0	1	3	4	5	6	7	8	10	18	20	21			
	20	4	90	96	60	48	56	48	180	36	40	42			
(31,97)	0	1	2	3	4	5	6	7	8	10	12	14			
	32	8	16	114	96	70	144	28	80	120	48	28			

(37,41)	0 6	1 3	3 36	4 84	5 60	8 24	9 36	11 66	12 18	14 42	16 24				
(37,43)	0 12	2 16	4 100	6 78	8 80	10 40	12 24	14 14	18 54						
(37,47)	0 12	2 8	4 20	6 66	8 16	10 70	12 24	14 42	16 48	18 36	22 22	42 42	50 50		
(37,53)	0 6	1 3	2 36	3 18	4 138	6 102	8 24	10 150	12 36						
(37,59)	0 12	4 12	6 24	8 8	10 40	12 12	14 14	16 96	18 36	22 110	26 52	30 30	60 60	64 64	
(37,61)	0 36	1 19	2 36	3 36	4 132	8 168	10 120	14 42							
(37,67)	0 16	2 48	4 132	6 192	8 32	10 80	12 24	14 14	16 48	20 60					
(37,71)	0 12	4 52	6 72	8 96	10 60	12 72	14 56	16 16	18 90	24 24	26 26	32 64	44 44		
(37,73)	0 30	1 7	2 24	3 96	4 102	5 60	6 168	8 168	12 18	15 30					
(37,79)	0 24	2 20	4 148	6 72	8 32	10 90	12 60	14 14	16 64	18 72	20 40	22 44	24 48	32 32	
(37,83)	0 20	4 20	6 96	8 40	10 120	14 168	18 18	24 48	28 56	30 60	34 34	46 46	72 72		
(37,89)	0 48	1 3	3 66	4 24	5 90	6 102	7 126	8 72	10 90	12 60	14 42	15 60	20 30	21 42	
(37,97)	0 24	1 7	2 48	3 96	4 282	5 30	6 306	15 30	18 36	20 30	28 42				

(41,43)	0 4	1 2	3 24	4 40	5 40	6 78	7 28	8 16	9 18	10 10	11 44	17 34	19 38	21 42	22 44
(41,47)	0 12	2 8	3 6	4 56	5 40	6 96	7 14	8 16	10 80	12 48	14 56	18 72			
(41,53)	0 12	1 3	2 12	3 48	4 96	6 36	9 18	10 30	12 168	15 30	16 48	18 36	20 30		
(41,59)	0 20	1 6	2 28	3 24	4 120	5 20	6 126	8 48	10 60	14 56	16 64	18 18	20 40		
(41,61)	0 24	1 3	2 24	3 6	4 36	5 30	6 96	8 24	9 54	10 60	12 72	15 30	16 96	32 96	
(41,67)	0 20	1 2	3 30	4 40	5 60	8 16	9 18	10 60	13 26	14 28	16 32	17 68	18 90	19 76	21 84
	32 64														
(41,71)	0 24	2 32	3 6	4 80	5 60	6 96	8 96	9 36	10 140	11 22	12 24	14 28	16 32	18 36	22 44
(41,73)	0 12	1 9	2 36	3 126	4 96	5 30	6 156	7 42	8 84	10 90	12 18	13 78			
(41,79)	0 12	3 96	4 32	5 50	6 60	7 56	8 48	9 18	10 100	11 44	12 24	13 52	18 72	20 40	22 44
	23 92														
(41,83)	0 24	1 2	2 52	3 18	4 64	5 10	6 192	8 64	10 110	12 48	13 52	14 98	16 32	18 36	20 80
(41,89)	0 12	1 21	2 120	3 90	4 234	6 168	8 120	10 60	12 36	14 84					
(41,97)	0 48	1 9	2 48	3 156	4 120	5 120	6 36	7 42	8 156	9 54	10 60	12 24	17 102	18 54	

(43,47)	0	4	6	8	12	18	20	24	28						
	6	42	36	132	60	18	120	72	42						
(43,53)	0	2	4	6	8	10	14	16	22	24	28	36			
	12	8	76	84	24	110	56	32	44	48	28	72			
(43,59)	0	1	2	3	4	5	6	7	8	9	10	12	14	16	18
	6	10	44	6	30	10	30	14	108	18	20	54	70	72	36
	30	34													
	30	102													
(43,61)	0	2	4	6	8	10	12	14	16	22	46				
	30	28	36	180	16	130	48	70	32	66	46				
(43,67)	0	1	2	3	4	5	6	8	10	12					
	24	14	80	42	198	10	162	72	110	36					
(43,71)	0	4	6	8	16	18	20	28	30	32	54	80			
	24	42	30	36	96	54	60	168	60	48	54	120			
(43,73)	0	1	2	3	4	5	6	8	9	10	11	14			
	16	6	40	78	152	20	120	48	18	210	22	84			
(43,79)	0	2	4	6	8	12	14	16	18	30	44				
	30	28	198	132	144	60	84	72	36	30	66				
(43,83)	0	1	2	3	4	5	6	8	10	12					
	36	10	56	18	270	30	222	36	150	96					
(43,89)	0	1	3	4	5	6	8	9	11	12	13	14	15	16	20
	12	2	66	48	10	60	16	36	22	24	104	56	30	128	40
	21	27	30	33	38										
	42	54	60	66	114										
(43,97)	0	1	2	3	4	5	6	7	8	9	10	12	13	14	25
	32	6	36	90	112	60	96	84	128	18	160	72	78	56	50
(47,53)	0	2	4	6	8	10	12	14	16	18	24				
	8	20	164	96	144	50	36	14	32	36	48				
(47,59)	0	2	4	6	8	10	12	16	20	24					
	24	12	138	78	120	90	168	24	30	36					
(47,61)	0	2	4	6	8	10	14	16	18	20	24	26	28	34	38
	8	4	44	24	56	20	56	48	90	60	24	26	28	34	76
	40	52	54												
	40	52	54												
(47,67)	0	4	6	8	10	12	16	18	20	40	60				
	12	48	54	72	90	108	144	108	30	60	90				
(47,71)	0	1	2	3	4	6	7	8	10	14	20				
	24	12	72	12	234	204	14	72	120	70	30				
(47,73)	0	2	3	4	5	6	7	8	9	10	11	14	15	16	18
	12	8	24	152	20	48	56	48	54	100	44	56	30	32	36
	26	28	30												
	52	56	60												
(47,79)	0	2	3	4	5	6	7	8	9	10	12	14	16	18	20
	24	24	18	96	10	126	28	60	18	80	54	56	48	90	150
	26														
	78														
(47,83)	2	4	6	8	12	16	18	20							
	48	168	228	48	324	96	36	60							
(47,89)	0	1	2	3	4	6	7	8	10	11	12	13	14	16	22
	32	4	24	12	80	96	14	112	120	132	120	26	168	96	44
(47,97)	0	1	2	3	4	5	6	7	8	9	10	12	18	19	22
	40	4	8	30	96	20	96	56	64	54	180	168	108	76	44
	28	38													
	56	76													

(53,59)	0	2	4	6	8	10	12	14	16	18	24	26	34		
	12	12	24	78	80	160	24	98	80	108	48	52	34		
(53,61)	0	1	3	4	6	8	9	10	12	14	24	26	36		
	18	15	24	96	18	48	18	60	72	210	72	78	108		
(53,67)	0	2	4	6	8	10	12	14	18	22	28	32	38	48	
	8	8	200	126	96	140	48	28	36	44	28	32	76	48	
(53,71)	0	2	4	6	8	10	12	14	16	18	20	22	24	34	
	30	24	124	96	128	60	120	112	16	36	100	44	48	34	
(53,73)	0	1	2	3	4	5	6	8	10	12	20	21	22	32	
	24	3	12	90	42	180	72	144	30	48	90	84	132	48	
(53,79)	0	4	6	8	10	12	14	16	20	22	24	26	28	30	32
	20	44	24	120	180	48	28	16	40	44	48	26	56	60	32
	34	38	54	58	76										
	68	38	54	58	76										
(53,83)	0	2	4	6	8	10	12	14	16	18	22	24	54		
	32	60	256	186	56	140	36	84	48	90	44	48	54		
(53,89)	0	1	2	3	4	5	6	7	9	12	16	20	24	30	
	54	3	24	60	126	30	228	42	18	252	96	60	132	90	
(53,97)	1	3	4	6	8	9	10	12	13	14	16	19	20	24	33
	3	48	54	192	96	54	240	144	78	42	48	114	60	84	66
(59,61)	0	2	4	6	8	10	12	14	18	20	22	26	36	38	40
	6	8	128	24	48	80	36	84	54	20	22	78	72	38	40
	42	48	54												
	42	96	54												
(59,67)	0	1	2	3	4	5	6	7	8	10	11	12	14	16	20
	18	6	12	24	192	30	138	14	12	90	44	54	154	48	30
	22	24	30												
	88	36	30												
(59,71)	0	4	6	8	12	16	18	20	40						
	12	138	126	168	318	120	108	30	60						
(59,73)	0	1	2	3	4	5	6	8	9	10	12	13	14	16	18
	12	2	8	48	24	70	60	16	36	40	96	156	14	32	72
	19	20	22	23	25	26	28	29							
	38	80	44	46	50	52	56	58							
(59,79)	2	4	6	8	12	16	18	20	22	28	30	32	40	42	100
	12	60	72	84	108	48	36	120	66	126	30	144	60	84	150
(59,83)	0	1	2	4	6	8	10	11	12	14	22				
	48	14	100	258	318	156	60	22	72	168	44				
(59,89)	0	1	2	3	4	5	6	7	8	10	14	16	18	19	21
	40	6	68	42	296	60	300	28	128	160	42	64	36	38	42
(59,97)	0	1	3	4	5	6	7	8	11	12	13	14	15	16	17
	16	2	54	128	90	60	28	136	22	72	130	56	90	96	34
	19	20	21	24	26	31									
	76	80	42	144	52	62									
(61,67)	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28
	12	12	60	42	64	30	72	112	32	54	60	88	48	130	56
	30	32													
	150	32													
(61,71)	0	2	4	6	8	10	12	14	16	18	20	22	26	30	32
	12	12	64	66	16	40	24	140	16	144	80	44	78	60	32
	42	48	58	64	76										
	42	48	58	64	76										
(61,73)	1	2	3	4	6	7	8	9	10	12	14	16	28		
	7	24	168	102	120	42	240	216	30	24	84	48	42		

(61,79)	0	2	4	6	8	10	12	14	16	18	20	24	26	44	
	30	20	228	180	232	140	72	112	32	36	40	48	26	44	
(61,83)	0	2	4	6	8	10	12	14	18	22	24	26	32	36	38
	20	4	72	30	104	120	60	70	54	110	24	52	64	108	76
	40	44	46	52	54	98									
	40	44	46	52	54	98									
(61,89)	0	1	2	3	4	5	6	8	10	14	15	16	18	24	50
	12	3	12	84	108	180	96	228	90	42	30	144	144	72	150
(61,97)	0	1	2	3	4	5	6	8	9	11	12	14	16	18	36
	90	7	24	36	204	60	84	48	72	132	288	168	216	36	54
(67,71)	0	4	6	8	10	12	14	18	20	24	28	40	48	64	
	6	66	30	144	30	156	84	144	150	72	42	60	144	96	
(67,73)	0	1	2	3	4	5	6	7	8	9	10	11	12	14	15
	16	6	8	96	160	140	54	70	192	18	120	22	48	126	60
	18	20	23												
	36	40	46												
(67,79)	0	2	4	6	8	10	12	16	18	20	24	48			
	36	4	138	378	144	90	36	144	180	30	108	72			
(67,83)	0	1	2	3	4	5	6	7	8	10	12	14	16	20	22
	6	6	36	12	204	10	222	14	96	170	108	154	168	60	132
	30														
	30														
(67,89)	0	1	2	3	4	5	6	7	8	9	11	12	13	14	15
	16	2	8	138	128	60	24	98	176	18	44	48	26	98	30
	16	17	18	19	27	37	40	42	44						
	64	102	72	38	54	74	40	84	88						
(67,97)	0	1	2	3	4	5	6	7	8	9	10	12	13	14	18
	32	6	56	102	168	60	240	56	208	144	40	48	52	112	162
	20	22	24												
	40	44	96												
(71,73)	0	3	4	5	6	7	8	10	11	13	14	15	16	19	20
	4	66	40	90	120	42	32	100	44	156	28	30	64	76	40
	22	23	24	28	31	34	38								
	44	46	48	56	62	68	76								
(71,79)	0	1	2	3	4	5	6	8	10	12	14	19	22	28	38
	12	8	40	6	168	70	150	312	170	90	42	38	132	126	76
(71,83)	0	2	4	6	8	12	14	16	18	20	28	32	36	54	
	24	12	162	60	108	144	168	168	36	150	252	48	126	54	
(71,89)	0	1	2	3	4	5	6	7	8	10	11	12	13	14	18
	20	4	40	36	120	60	276	28	192	180	22	144	78	28	108
	20	22	34												
	40	176	68												
(71,97)	0	1	3	4	5	6	7	8	9	10	11	13	14	15	17
	28	4	240	40	70	72	84	112	108	140	44	52	56	30	34
	20	22	23	24	30	44	45								
	80	132	92	48	120	88	90								
(73,79)	0	1	2	3	4	5	7	8	9	10	11	12	14	15	16
	8	4	8	156	136	60	70	96	18	200	132	72	168	150	32
	18	23	26												
	72	46	52												
(73,83)	0	1	2	4	5	6	7	8	9	10	11	12	14	18	20
	12	2	8	184	50	66	28	80	54	130	22	168	84	72	80
	26	29	30	34	54	55									
	104	58	120	68	54	110									

(73,89)	0	1	2	3	4	5	6	7	8	9	10	12	14	17	18
	12	9	24	240	156	150	96	42	120	54	120	96	126	102	108
	20	25													
	60	150													
(73,97)	0	1	2	3	4	5	6	7	8	10	12	14			
	24	37	108	108	342	120	396	126	156	240	72	84			
(79,83)	4	6	8	10	12	16	18	20	24	26	28	36	40	42	44
	42	144	72	60	36	48	180	30	72	156	126	54	180	126	66
	56	76	90												
	84	114	90												
(79,89)	0	2	3	4	5	6	7	8	9	10	11	12	13	15	16
	16	8	150	96	20	48	42	32	72	80	44	24	26	30	64
	17	18	19	21	22	23	26	44	47						
	68	108	114	210	176	138	52	88	94						
(79,97)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	24	4	32	66	72	30	120	70	80	162	20	198	144	156	168
	15	16	17	18	19	22	23	24	26						
	30	160	68	36	38	44	138	48	52						
(83,89)	0	1	2	3	4	6	7	8	10	11	12	14	15	16	18
	8	2	24	24	128	318	14	32	240	88	144	196	60	96	108
	22	24	26	28	38										
	176	48	52	56	76										
(83,97)	0	1	2	4	5	6	8	9	10	12	13	14	16	18	20
	12	2	8	112	30	12	136	72	40	120	52	112	224	72	120
	21	22	25	26	30	31	34	38	39	56	58	62			
	42	44	50	104	60	62	68	76	78	168	58	124			
(89,97)	0	1	3	4	5	6	7	8	10	12	13	14	17	20	24
	24	21	312	534	60	138	168	48	180	24	78	252	102	180	84

If $p \neq 3$, then there is always a period dividing 4 in the list above. Therefore, we get the following corollary.

Corollary 1. *Let (p, l) be a pair of prime numbers such that $3 < p < l < 100$. Then there are $x, y \in \tilde{\Gamma}_{p,l}$ such that $xy = yx$ and $|x|^2 = p^4, |y|^2 = l^4$.*

Note that $\mathrm{per}(a, b) = 1$ if and only if $ab = ba \in \Gamma_{p,l}$.

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