ABELIAN SUBGROUP STRUCTURE OF SQUARE COMPLEX GROUPS AND ARITHMETIC OF QUATERNIONS

DIEGO RATTAGGI AND GUYAN ROBERTSON

1. Introduction

A square complex is a 2-complex formed by gluing squares together. This article is concerned with the fundamental group Γ of certain square complexes of nonpositive curvature, related to quaternion algebras. The abelian subgroup structure of Γ is studied in some detail. Before outlining the results, it is necessary to describe the construction of Γ.

In [Moz, Section 3], there is constructed a lattice subgroup $\Gamma = \Gamma_{p,l}$ of $G = PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$, where $p, l \equiv 1 \pmod 4$ are two distinct primes. This restriction was made because -1 has a square root in $\mathbb{Q}_p$ if and only if $p \equiv 1 \pmod 4$, but the construction of Γ is generalized in [Rat, Chapter 3] to all pairs (p, l) of distinct odd primes.

The affine building Δ of G is a product of two homogeneous trees of degrees $(p+1)$ and $(l+1)$ respectively. The group Γ is a finitely presented torsion free group which acts freely and transitively on the vertices of Δ, with a finite square complex as quotient Δ/Γ.

Here is how Γ is constructed. Let

$$\mathbb{H}(\mathbb{Z}) = \{x = x_0 + x_1 i + x_2 j + x_3 k; x_0, x_1, x_2, x_3 \in \mathbb{Z}\}$$

be the ring of integer quaternions where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$. Let $\overline{x} = x_0 - x_1 i - x_2 j - x_3 k$ be the conjugate of x, and $|x|^2 = x\overline{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2$ its norm.

Let $c_p, d_p \in \mathbb{Q}_p$ and $c_l, d_l \in \mathbb{Q}_l$ be elements such that $c_p^2 + d_p^2 + 1 = 0$, $c_l^2 + d_l^2 + 1 = 0$. Such elements exist by Hensel's Lemma and [DSV, Proposition 2.5.3]. We can take $d_p = 0$, if $p \equiv 1 \pmod 4$, and $d_l = 0$, if $l \equiv 1 \pmod 4$. Define

$$\psi : \mathbb{H}(\mathbb{Z}) - \{0\} \to PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$$

Date: November 30, 2004.

2000 *Mathematics Subject Classification.* Primary: 11R52, 52C20, 20E08, 20E07. Secondary: 20E42.

by

$$\psi(x) = \left(\begin{pmatrix} x_0 + x_1c_p + x_3d_p & -x_1d_p + x_2 + x_3c_p \\ -x_1d_p - x_2 + x_3c_p & x_0 - x_1c_p - x_3d_p \end{pmatrix}, \begin{pmatrix} x_0 + x_1c_l + x_3d_l & -x_1d_l + x_2 + x_3c_l \\ -x_1d_l - x_2 + x_3c_l & x_0 - x_1c_l - x_3d_l \end{pmatrix} \right). \tag{1}$$

This formula abuses notation by identifying an element of $PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$ with its representative in $GL_2(\mathbb{Q}_p) \times GL_2(\mathbb{Q}_l)$.

Note that $\psi(xy) = \psi(x)\psi(y)$, $\psi(\lambda x) = \psi(x)$, if $\lambda \in \mathbb{Z} - \{0\}$, and $\psi(x)^{-1} = \psi(\overline{x})$. Moreover the inverse image under ψ of the identity element in $PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$ is precisely

$$\mathbb{Z} - \{0\} = \{x \in \mathbb{H}(\mathbb{Z}); x_0 \neq 0, x_1 = x_2 = x_3 = 0\}.$$

Let

$$\begin{aligned} \tilde{\Gamma} = \{x \in \mathbb{H}(\mathbb{Z})\,; \quad & |x|^2 = p^r l^s, r, s \geq 0\,; \\ & x_0 \text{ odd}, x_1, x_2, x_3 \text{ even, if } |x|^2 \equiv 1 \pmod 4\,; \\ & x_1 \text{ even}, x_0, x_2, x_3 \text{ odd, if } |x|^2 \equiv 3 \pmod 4\}. \end{aligned}$$

Then $\Gamma = \psi(\tilde{\Gamma})$ is a torsion free cocompact lattice in G. Let

$$\tilde{A} = \{x \in \tilde{\Gamma}; x_0 > 0, |x|^2 = p\}, \quad \tilde{B} = \{y \in \tilde{\Gamma}; y_0 > 0, |y|^2 = l\}.$$

Then $\tilde{A}$ contains $p+1$ elements and $\tilde{B}$ contains $l+1$ elements, by a result of Jacobi [Lub, Theorem 2.1.8]. The images $A = \psi(\tilde{A}), B = \psi(\tilde{B})$ of $\tilde{A}, \tilde{B}$ in Γ generate free groups $\Gamma_p = \langle A \rangle = \langle a_1, \dots, a_{\frac{p+1}{2}} \rangle$, $\Gamma_l = \langle B \rangle = \langle b_1, \dots, b_{\frac{l+1}{2}} \rangle$ of ranks $(p+1)/2$, $(l+1)/2$ respectively and Γ itself is generated by $A \cup B$. The 1-skeleton of Δ is the Cayley graph of Γ relative to this set of generators. The group Γ has a finite presentation with generators $\{a_1, \dots, a_{\frac{p+1}{2}}\} \cup \{b_1, \dots, b_{\frac{l+1}{2}}\}$ and $(p+1)(l+1)/4$ relations of the form $ab = \tilde{b}\tilde{a}$, where $a, \tilde{a} \in A$, $b, \tilde{b} \in B$. In fact, given any $a \in A$, $b \in B$, there are unique elements $\tilde{a} \in A$, $\tilde{b} \in B$ such that $ab = \tilde{b}\tilde{a}$. This follows from a special case of Dickson's factorization property for integer quaternions ([Dic, Theorem 8]).

Proposition 1.1. *(*[Dic]*) Let $x \in \tilde{\Gamma}$ such that $|x|^2 = pl$. Then there are uniquely determined $z, \tilde{z} \in \tilde{A}$, $y, \tilde{y} \in \tilde{B}$ such that $zy, \tilde{y}\tilde{z} = \pm x$.*

It is worth noting that $zy \neq \tilde{y}\tilde{z}$ in general, as demonstrated by the following example.

Example 1.2. Let $p = 3$, $l = 5$ and $x = 1 + 2i + j + 3k$. Then $(1 - j + k)(1 + 2i) = x$ and $(1 - 2k)(1 - j - k) = -x$.

We can now outline the contents of this article. A fundamental fact, upon which much else depends, is that Γ is *commutative transitive*, in the sense that the relation of commutativity is transitive on non-trivial elements of Γ. In particular Γ cannot contain a subgroup isomorphic to $F_2 \times F_2$, where F_2 denotes the free group of rank 2. Furthermore, Γ is a *CSA-group*, i.e. all its maximal abelian subgroups Γ_0 satisfy $g\Gamma_0 g^{-1} \cap \Gamma_0 = \{1\}$ for all $g \in \Gamma - \Gamma_0$.

Every nontrivial element $\gamma \in \Gamma$ is the image under ψ of a quaternion of the form $x_0 + z_0(c_1 i + c_2 j + c_3 k)$ where $c_1, c_2, c_3 \in \mathbb{Z}$ are relatively prime. The element γ is contained in a unique maximal abelian subgroup Γ_0 and the integer $n = n(\Gamma_0) = c_1^2 + c_2^2 + c_3^2$ depends only on Γ_0 rather than the particular choice of γ. We define a class of maximal abelian subgroups of Γ isomorphic to $\mathbb{Z}^2$, which we call period subgroups, and which are characterized by the condition $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$. Every maximal abelian subgroup $\Gamma_0 \cong \mathbb{Z}^2$ is conjugate in Γ to a period subgroup and, as the name suggests, period subgroups are closely related to periodic tilings of the plane. On the other hand, some maximal abelian subgroups of Γ are isomorphic to $\mathbb{Z}$, and we show how to construct these. Several explicit examples and counterexamples are included.

2. The CSA property

Let $\tau : \mathbb{H}(\mathbb{Q}) - \mathbb{Q} \to \mathbb{P}^2(\mathbb{Q})$ be defined by $\tau(x) = \mathbb{Q}(x_1, x_2, x_3)$, which is a line in $\mathbb{Q}^3$ through $(0, 0, 0)$. By [Moz, Section 3] , two quaternions $x, y \in \mathbb{H}(\mathbb{Q}) - \mathbb{Q}$ commute if and only if $\tau(x) = \tau(y)$. This directly implies the following lemma, which in turn has Proposition 2.2 as a consequence, see also [Rat, Chapter 3].

Lemma 2.1. *Elements $x, y \in \tilde{\Gamma}$ commute if and only if their images $\psi(x), \psi(y) \in \Gamma$ commute.*

A group is said to be *commutative transitive* if the relation of commutativity is transitive on its non-trivial elements.

Proposition 2.2. *The group Γ is commutative transitive.*

Wise has asked in [Wis, Problem 10.9] whether the fundamental group of any nonelementary complete square complex contains a subgroup isomorphic to $F_2 \times F_2$. We can give a negative answer of this question, since our group Γ belongs to this class of fundamental groups, and it is a direct consequence of Proposition 2.2 that Γ does not contain a $F_2 \times F_2$ subgroup. In fact, since Γ is torsion free, and a (free) abelian subgroup of Γ has rank ≤ 2 [Pra, Lemma 3.2], we have a more precise result.

Corollary 2.3. *The only nontrivial direct product subgroup of Γ is $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.*

If $\gamma = \psi(x) \in \Gamma - \{1\}$ then the centralizer $\Gamma_0 = Z_\Gamma(\gamma)$ is the unique maximal abelian subgroup of Γ containing γ. Moreover Γ_0 is determined by $\tau(x)$, independent of the choice of x.

As described in [MR, Remark 4], a group is commutative transitive if and only if the centralizer of any non-trivial element is abelian. A third equivalent condition (called *SA-property* in [MR]) is proved for Γ in the following lemma. It is used to show in Proposition 2.6 that Γ is a *CSA-group*, i.e. all its maximal abelian subgroups are malnormal, where a subgroup Γ_0 of Γ is *malnormal* (or *conjugate separated*) if $g\Gamma_0 g^{-1} \cap \Gamma_0 = \{1\}$ for all $g \in \Gamma - \Gamma_0$. Any CSA-group is commutative transitive, but the converse is not true, see [MR].

Lemma 2.4. *If Γ_1 and Γ_2 are maximal abelian subgroups of Γ and $\Gamma_1 \neq \Gamma_2$ then $\Gamma_1 \cap \Gamma_2 = \{1\}$.*

Proof. Suppose that there exists a nontrivial element $\gamma \in \Gamma_1 \cap \Gamma_2$. If $\gamma_i \in \Gamma_i - \{1\}$, $i = 1, 2$, then $\gamma\gamma_1 = \gamma_1\gamma$ and $\gamma\gamma_2 = \gamma_2\gamma$ which implies $\gamma_1\gamma_2 = \gamma_2\gamma_1$ by Proposition 2.2. Since Γ_1, Γ_2 are maximal abelian, $\Gamma_1 = \Gamma_2$. □

It is well known that there is a (surjective) homomorphism

$$\theta : \mathbb{H}(\mathbb{Q}) - \{0\} \to \mathrm{SO}_3(\mathbb{Q})$$

defined by $\theta(y)x = yxy^{-1}$, for $x = x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Q})$ identified with $(x_1, x_2, x_3) \in \mathbb{Q}^3$.

If $y \in \mathbb{H}(\mathbb{Q}) - \mathbb{Q}$ then the axis of rotation of $\theta(y)$ is $\tau(y)$. This is an immediate consequence of the fact that

$$\theta(y)(y - y_0) = y(y - y_0)y^{-1} = y - y_0 \,.$$

Moreover the angle of rotation is 2α where $\cos\alpha = \frac{y_0}{|y|}$ [Vig, Chapitre I, §3]. In particular, the angle of rotation is a multiple of π only if $y_0 = 0$.

Lemma 2.5. *(a) Suppose that $x, y \in \mathbb{H}(\mathbb{Q}) - \mathbb{Q}$ and $y_0 \neq 0$. Then yxy^{-1} commutes with x if and only if y commutes with x.*

(b) If $a, b \in \Gamma$, then bab^{-1} commutes with a if and only if b commutes with a.

Proof. (a) If yxy^{-1} commutes with x, then the rotations $\theta(yxy^{-1})$ and $\theta(x)$ have the same axis. However, the axis of $\theta(yxy^{-1}) = \theta(y)\theta(x)\theta(y)^{-1}$ is $\theta(y)\tau(x)$. Therefore $\theta(y)\tau(x) = \tau(x)$: in other words $\theta(y)(x_1, x_2, x_3) = \pm(x_1, x_2, x_3)$. Now if $\theta(y)(x_1, x_2, x_3) = -(x_1, x_2, x_3)$ then $\theta(y)$ is a rotation of angle π, with axis perpendicular to (x_1, x_2, x_3). This cannot happen since $y_0 \neq 0$. Therefore $\theta(y)$ has axis $\tau(x)$. That is $\tau(y) = \tau(x)$, and consequently y commutes with x. The converse is clear.

(b) If $a = 1$ or $b = 1$, the statement is obvious. If $a, b \in \Gamma - \{1\}$ and bab^{-1} commutes with a, then representatives x, y for a, b in $\mathbb{H}(\mathbb{Q}) - \mathbb{Q}$ have nonzero real parts and satisfy the same relation, by Lemma 2.1. The assertion follows from (a). Again, the converse is clear. □

Proposition 2.6. Γ *is CSA.*

Proof. Suppose that Γ_0 is a maximal abelian subgroup of Γ and that $b \in \Gamma$, with $b\Gamma_0 b^{-1} \cap \Gamma_0 \neq \{1\}$. We must show that $b \in \Gamma_0$.

By Lemma 2.4, $b\Gamma_0 b^{-1} = \Gamma_0$. Let $a \in \Gamma_0$. Then bab^{-1} commutes with a and so, by Lemma 2.5, b commutes with a. Since Γ_0 is maximal abelian, $b \in \Gamma_0$. □

We now recall the following known result.

Lemma 2.7. *(a) (*[MR, Proposition 9(5)]*) A non-abelian CSA-group has no non-abelian solvable subgroups.*

(b) ([MR, Proposition 10(3)]*) Subgroups of CSA-groups are CSA.*

Corollary 2.8. *Let $a \in \Gamma_p - \{1\}$ and $b \in \Gamma_l - \{1\}$. Then either $\langle a, b\rangle \cong \mathbb{Z}^2$ or $\langle a, b\rangle$ contains a free subgroup of rank 2.*

Proof. If a, b commute, then $\langle a, b\rangle \cong \mathbb{Z}^2$, since Γ is torsion free and $\langle a, b\rangle$ is not cyclic. Assume that a, b do not commute. We will show that $\langle a, b\rangle$ is not virtually solvable. The Tits Alternative for finitely generated linear groups (see [Tit]) then implies that $\langle a, b\rangle$ contains a free subgroup of rank 2. Note that Γ is linear, see [Rat, Section 3.2] for an explicit injective homomorphism $\Gamma \to SO_3(\mathbb{Q})$. Let U be a finite index subgroup of $\langle a, b\rangle$, in particular there are $r, s \in \mathbb{N}$ such that $a^r, b^s \in U$. The elements a^r and b^s do not commute since otherwise also a and b would commute by Proposition 2.2. It follows that U is not abelian. By Proposition 2.6 and Lemma 2.7(b), $\langle a, b\rangle$ is CSA. Lemma 2.7(a) shows that U is not solvable. □

3. Maximal abelian subgroups and period subgroups.

Recall that the group Γ acts freely and transitively on the vertex set of the affine building Δ of $PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$. The building Δ is a product of two homogeneous trees and the apartments (maximal flats) in Δ are copies of the Euclidean plane tesselated by squares.

Notation 3.1. If n is an integer and p is an odd prime, then the **Legendre symbol** is

$$\left(\frac{n}{p}\right) = \begin{cases} 0 & \text{if } p \mid n, \\ 1 & \text{if } p \nmid n \text{ and } n \text{ is a square mod } p, \\ -1 & \text{if } p \nmid n \text{ and } n \text{ is not a square mod } p. \end{cases}$$

Any element of $\Gamma - \{1\}$ is the image under ψ of a quaternion of the form

$$x = x_0 + z_0(c_1 i + c_2 j + c_3 k)\,, \qquad (2)$$

where $c_1, c_2, c_3 \in \mathbb{Z}$ are relatively prime, $z_0 \neq 0$, $(c_1, c_2, c_3) \neq (0,0,0)$, and

$$|x|^2 = x_0^2 + (c_1^2 + c_2^2 + c_3^2)z_0^2 = p^r l^s, r, s \geq 0\,.$$

Recall that $\tau(x) = \mathbb{Q}(c_1, c_2, c_3) \in \mathbb{P}^2(\mathbb{Q})$ and recall that elements $\psi(x), \psi(y) \in \Gamma - \{1\}$ commute if and only if $\tau(x) = \tau(y)$. Moreover the centralizer $\Gamma_0 = Z_\Gamma(\psi(x))$ is the unique maximal abelian subgroup of Γ containing $\psi(x)$. Let

$$n(x) = n(\psi(x)) = n(\Gamma_0) = c_1^2 + c_2^2 + c_3^2\,.$$

An abelian subgroup of Γ has rank ≤ 2 [Pra, Lemma 3.2]. Since Γ is torsion free, a nontrivial abelian subgroup Γ_0 of Γ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}^2$. If $\Gamma_0 \cong \mathbb{Z}^2$ then there is a unique apartment $\mathcal{A}_{\Gamma_0}$ which is stabilized by Γ_0 [Pra, 6.8], and Γ_0 acts cocompactly by translation on this apartment. We call $\mathcal{A}_{\Gamma_0}$ a *periodic* apartment.

Definition 3.2. A maximal abelian subgroup $\Gamma_0 \cong \mathbb{Z}^2$ will be called a **period subgroup** if the apartment $\mathcal{A}_{\Gamma_0}$ contains the vertex O of Δ whose stabilizer in G is $PGL_2(\mathbb{Z}_p) \times PGL_2(\mathbb{Z}_l)$.

Since the action of Γ on Δ is vertex transitive, every maximal abelian subgroup $\Gamma_0 \cong \mathbb{Z}^2$ is conjugate in Γ to a period subgroup. We want to show that $n(x)$ determines when $Z_\Gamma(\psi(x))$ is a period subgroup of Γ.

Recall that Γ is generated by free groups Γ_p, Γ_l, of ranks $(p+1)/2$, $(l+1)/2$ respectively. If $\gamma \in \Gamma$, let $\ell(\gamma)$ denote the natural word length of γ, in terms of the generators of Γ_p, Γ_l. The condition $\ell(\gamma^2) = 2\ell(\gamma)$, which is used in the next lemma, is equivalent to the assertion that γ has an axis containing O, upon which γ acts by translation.

Lemma 3.3. *Let $a = \psi(x) \in \Gamma_p - \{1\}$ and let $n = n(x)$. The following statements are equivalent.*

(a) $p \nmid n$;
(b) $\ell(a^2) = 2\ell(a)$;
(c) $\left(\frac{-n}{p}\right) = 1$.

Similar equivalent assertions hold, if p is replaced by l.

Before giving the proof, we note that

$$\left(\frac{-n}{p}\right) = \begin{cases} \left(\frac{n}{p}\right), & \text{if } p \equiv 1 \pmod 4\,, \\ -\left(\frac{n}{p}\right), & \text{if } p \equiv 3 \pmod 4\,. \end{cases}$$

Proof. (*a*) $\Leftrightarrow$ (*b*). The idea for this comes from the proof of [Moz, Proposition 3.15]. Write x as in (2) with $|x|^2 = x_0^2 + nz_0^2 = p^r$, $r > 0$. Extracting a common factor, if necessary, we may assume $\gcd(x_0, z_0) = 1$. This means that $r = \ell(a)$ [Rat, Corollary 3.11(4), Theorem 3.30(1)].

Suppose that $p \nmid n$. To prove $\ell(a^2) = 2\ell(a)$ we must show that p does not divide x^2. Now if p divides

$$x^2 = (x_0^2 - nz_0^2) + 2x_0z_0(c_1 i + c_2 j + c_3 k)\,,$$

then p divides the real part $x_0^2 - nz_0^2$. Therefore p divides x_0 (since p divides $p^r = x_0^2 + nz_0^2$). But this implies that p divides z_0, since $p \nmid n$. This contradicts the assumption $\gcd(x_0, z_0) = 1$.

Conversely, suppose that $\ell(a^2) = 2\ell(a)$. If p divides n, then p divides x_0 (since p divides $x_0^2 + nz_0^2$). Therefore p divides the real and imaginary parts of $x^2 = (x_0^2 - nz_0^2) + 2x_0z_0(c_1i + c_2j + c_3k)$. But this implies that $\ell(a^2) < 2r$, a contradiction.

$(a) \Leftrightarrow (c)$. Suppose that $p \nmid n$. Note that p does not divide z_0: otherwise p also divides x_0. It follows that z_0 has a multiplicative inverse $(\mathrm{mod}\ p)$. That is, one can choose $t \in \mathbb{Z}$ such that $z_0t \equiv 1 \pmod p$. Then

$$0 \equiv (x_0^2 + nz_0^2)t^2 \equiv x_0^2t^2 + n \pmod p\,.$$

Since $p \nmid n$, this means that $\left(\frac{-n}{p}\right) = 1$. The converse is obvious. □

Lemma 3.4. *If $\Gamma_0 \cong \mathbb{Z}^2$ is a period subgroup of Γ and $n = n(\Gamma_0)$, then $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$.*

Proof. The group Γ_0 acts cocompactly by translation on the apartment $\mathcal{A}_{\Gamma_0}$ containing the vertex O. It follows that Γ_0 contains elements $a \in \Gamma_p - \{1\}$, $b \in \Gamma_l - \{1\}$. These elements act freely by translation on the apartment, and so $\ell(a^2) = 2\ell(a)$, $\ell(b^2) = 2\ell(b)$. Therefore $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$, by Lemma 3.3. □

Lemma 3.5. *If $\gamma = \psi(x) \in \Gamma - (\Gamma_p \cup \Gamma_l)$ and* $\gcd(n(x), pl) = 1$*, then $Z_\Gamma(\gamma)$ is a period subgroup of Γ.*

Proof. Let $x = x_0 + z_0(c_1i + c_2j + c_3k)$ as in (2) and $n = n(x) = c_1^2 + c_2^2 + c_3^2$. We may assume $\gcd(x_0, z_0) = 1$ and $|x|^2 = x_0^2 + nz_0^2 = p^rl^s$, where $r, s \geq 1$ because $\psi(x) \notin \Gamma_p \cup \Gamma_l$.

The assumption $\gcd(n, pl) = 1$ implies that $\gcd(x_0z_0, pl) = 1$. For example, if $p \mid x_0$ then $p \mid z_0$, since $p \mid (x_0^2 + nz_0^2)$ and $p \nmid n$. This contradicts $\gcd(x_0, z_0) = 1$. Similarly $p \nmid z_0$. It follows from the "if" part of the proof of [Moz, Proposition 3.15] (and an obvious generalization to the cases where $p \equiv 3 \pmod 4$ or $l \equiv 3 \pmod 4$) that $\gamma = \psi(x)$ lies in an abelian subgroup Γ_0 of Γ, with $\Gamma_0 \cong \mathbb{Z}^2$. The same proof also shows that Γ_0 acts cocompactly by translation on an apartment $\mathcal{A}$ containing O. (The essential point in the proof of Mozes is that $\ell(\gamma^2) = 2\ell(\gamma)$.) However, $Z_\Gamma(\gamma)$ is the unique maximal abelian subgroup containing Γ_0. Therefore $Z_\Gamma(\gamma)$ acts cocompactly by translation on the apartment $\mathcal{A}$, by the uniqueness assertion in [Pra, 6.8]. In other words, $Z_\Gamma(\gamma)$ is a period subgroup of Γ. □

Now we can describe the period subgroups of Γ.

Proposition 3.6. *Let Γ_0 be a maximal abelian subgroup of Γ, and let $n = n(\Gamma_0)$. Then Γ_0 is a period subgroup if and only if $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$.*

Before proceeding with the proof, we introduce some notation. There is a canonical Cartan subgroup C of $G = PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$ defined by

$$C = \left(\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right) \cap G .$$

The group C acts by translation on an apartment $\mathcal{A}$, which contains the vertex O whose stabilizer in G is $PGL_2(\mathbb{Z}_p) \times PGL_2(\mathbb{Z}_l)$. The action of C is transitive on the vertices of $\mathcal{A}$.

Proof of Proposition 3.6. In view of Lemma 3.4, it suffices to show that $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$ implies that Γ_0 is a period subgroup. Suppose therefore that $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$. Then $\gcd(n, pl) = 1$. The result will therefore follow from Lemma 3.5, if we can show that Γ_0 is not contained in $\Gamma_p \cup \Gamma_l$. By symmetry it is enough to prove that if Γ_0 contains an element $b = \psi(y) \in \Gamma_l - \{1\}$, then it also contains an element $a = \psi(x) \in \Gamma_p - \{1\}$. For then the element ba does not lie in $\Gamma_p \cup \Gamma_l$.

Write $y = y_0 + z_0(c_1 i + c_2 j + c_3 k)$, where $c_1, c_2, c_3 \in \mathbb{Z}$ are relatively prime and $n = n(y) = c_1^2 + c_2^2 + c_3^2$. The quaternion y represents the element b of Γ_l of word length $\ell(b) = s > 0$. By Lemma 3.3, b acts by translation of distance s along an axis L_b containing O.

The element of $GL_2(\mathbb{Q}_p) \times GL_2(\mathbb{Q}_l)$ corresponding to y in the formula (1) has eigenvalues $y_0 \pm z_0\sqrt{-n}$. The assumption $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$ implies that $\sqrt{-n}$ exists in both $\mathbb{Q}_p$ and $\mathbb{Q}_l$ and therefore that b is diagonalizable in G. In other words, there exists an element $h \in G$ such that $h^{-1}bh \in C$.

The group hCh^{-1} acts by translation on the apartment $h\mathcal{A}$. Also the element $b \in hCh^{-1} \cap \Gamma_l$ acts by translation on the apartment $h\mathcal{A}$, in a direction which will be called "vertical". Now $h\mathcal{A}$ necessarily contains the axis L_b of b, by [BH, Theorem II.6.8 (3)]. In particular, $O \in h\mathcal{A}$.

Choose $g \in hCh^{-1}$ to act on $h\mathcal{A}$ by horizontal translation. Consider the horizontal strip H in $h\mathcal{A}$ obtained by translating the vertical segment $[O, bO]$.

Since Γ acts freely and transitively on the vertices of Δ, each vertical segment $g^i[O, bO]$ of H lies in the Γ-orbit of precisely one segment of the form $[O, \gamma O]$, $\gamma \in \Gamma_l$, $\ell(\gamma) = s$. Moreover, there are only finitely many such segments $[O, \gamma O]$.

If $i > 0$ then $g^i O = u_i O$, for some $u_i \in \Gamma_p - \{1\}$. Since b and g commute, we have $g^i bO = bg^i O = bu_i O$. That is, $g^i[O, bO] = [u_i O, bu_i O]$, which lies in the Γ-orbit of the segment $[O, u_i^{-1}bu_i O]$. By the finiteness

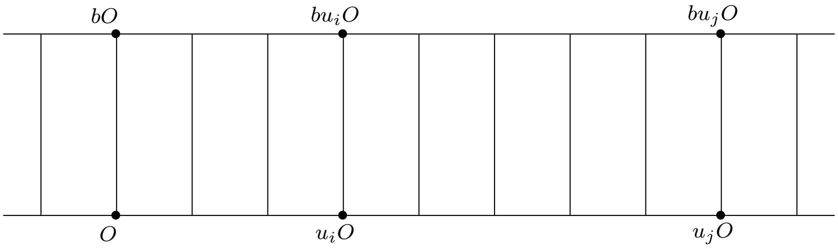

FIGURE 1. The horizontal strip H.

assertion in the preceding paragraph, there exist integers $j > i > 0$ such that

$$[O, u_i^{-1}bu_iO] = [O, u_j^{-1}bu_jO] .$$

By freeness of the action of Γ,

$$u_i^{-1}bu_i = u_j^{-1}bu_j ,$$

and $u_i \neq u_j$. Therefore $ab = ba$, where $a = u_iu_j^{-1} \in \Gamma_p - \{1\}$. □

A maximal abelian subgroup Γ_0 of Γ may be isomorphic to $\mathbb{Z}$. Here is a way of providing some examples.

Corollary 3.7. *Suppose that $a \in \Gamma_p - \{1\}$, and $n = n(a)$ satisfies*

$$\left(\frac{-n}{p}\right) = 1, \quad \left(\frac{-n}{l}\right) = -1 .$$

Then $Z_\Gamma(a) < \Gamma_p$ is a maximal abelian subgroup of Γ, and $Z_\Gamma(a) \cong \mathbb{Z}$. A similar assertion applies to elements of $\Gamma_l - \{1\}$.

Proof. The hypothesis implies that $\gcd(n, pl) = 1$. If $Z_\Gamma(a) \not\subset \Gamma_p$, then $Z_\Gamma(a)$ contains an element $\gamma \notin \Gamma_p \cup \Gamma_l$. Therefore $Z_\Gamma(a) = Z_\Gamma(\gamma)$ is a period group, by Lemma 3.5. But this implies $\left(\frac{-n}{l}\right) = 1$, by Proposition 3.6, – a contradiction. □

Example 3.8. Let $\Gamma = \Gamma_{3,5}$. This group has a presentation with generators $\{a_1, a_2, b_1, b_2, b_3\}$ and relators

$$\{a_1b_1a_2b_2,\ a_1b_2a_2b_1^{-1},\ a_1b_3a_2^{-1}b_1,\ a_1b_3^{-1}a_1b_2^{-1},\ a_1b_1^{-1}a_2^{-1}b_3,\ a_2b_3a_2b_2^{-1}\} ,$$

where

$$\begin{aligned} a_1 &= \psi(1 + j + k), & a_1^{-1} &= \psi(1 - j - k),\\ a_2 &= \psi(1 + j - k), & a_2^{-1} &= \psi(1 - j + k),\\ b_1 &= \psi(1 + 2i), & b_1^{-1} &= \psi(1 - 2i),\\ b_2 &= \psi(1 + 2j), & b_2^{-1} &= \psi(1 - 2j),\\ b_3 &= \psi(1 + 2k), & b_3^{-1} &= \psi(1 - 2k). \end{aligned}$$

The subgroup $\langle a_1 \rangle = Z_\Gamma(a_1) < \Gamma_3$ is maximal abelian in Γ by Corollary 3.7, since $n(a_1) = 2$, $\left(\frac{-2}{3}\right) = 1$ and $\left(\frac{-2}{5}\right) = -1$.

The subgroup $\langle a_1 a_2^{-1} a_1^2 \rangle = \langle \psi(-5 - 6i - 2j + 4k) \rangle$ is not maximal abelian. It is contained in the period subgroup

$$\Gamma_0 = \langle a_1 a_2^{-1} a_1^2,\, b_3 b_2^{-1} b_3^{-1} b_1 \rangle \cong \mathbb{Z}^2 .$$

Indeed, $n(\Gamma_0) = n(a_1 a_2^{-1} a_1^2) = 14$, $\left(\frac{-14}{3}\right) = 1$, $\left(\frac{-14}{5}\right) = 1$. Note that $b_3 b_2^{-1} b_3^{-1} b_1 = \psi(-11 + 18i + 6j - 12k)$. Part of the period lattice for Γ_0 is illustrated in Figure 2.

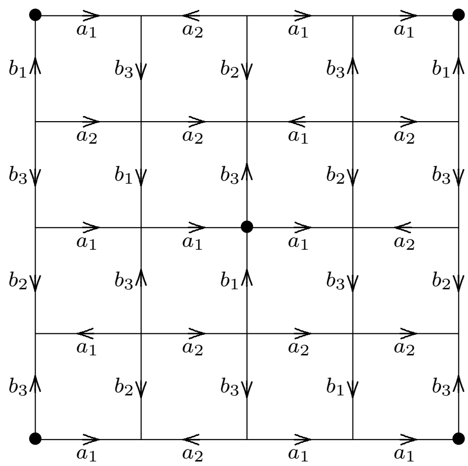

FIGURE 2. Part of a periodic apartment for $\Gamma_0 < \Gamma_{3,5}$.

Example 3.9. Let $\Gamma = \Gamma_{3,5}$. Consider $b_1 a_1 b_1^{-1} = \psi(5 - 7j + k)$. By Example 3.8, $\langle a_1 \rangle$ is maximal abelian in Γ. Therefore so also is $\Gamma_0 = \langle b_1 a_1 b_1^{-1} \rangle = b_1 \langle a_1 \rangle b_1^{-1}$. Now $\gamma = b_1 a_1^6 b_1^{-1} = a_2 a_1^{-1} a_2^{-2} a_1^{-1} a_2 = \psi(5(23 + 14j - 2k)) = \psi(x) \in \Gamma_3$, with $|x|^2 = 5^2.3^6$. Also $n(x) = n(\Gamma_0) = 50$, $\left(\frac{-50}{3}\right) = 1$ and $\left(\frac{-50}{5}\right) = 0$. There is a periodic horizontal strip of height 2 (Figure 3), upon which γ acts by translation. This strip is the union of the axes of γ.

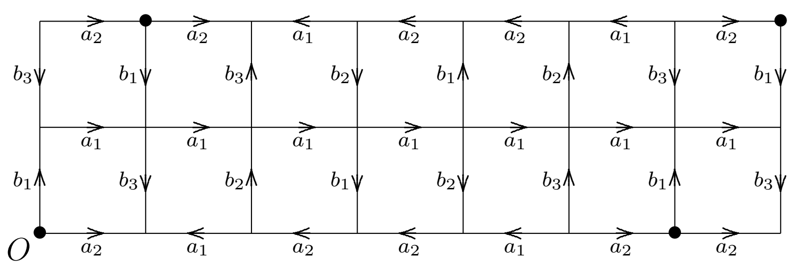

FIGURE 3. Part of a periodic horizontal strip .

Example 3.10. Let $\Gamma = \Gamma_{3,5}$. Conjugating the period subgroup $\langle a_1 a_2^{-1} a_1^2, b_3 b_2^{-1} b_3^{-1} b_1 \rangle$ of Example 3.8 by a_2 gives the group

$$\begin{aligned}\Gamma_0 &= \langle a_2 a_1 a_2^{-1} a_1^2 a_2^{-1}, a_2 b_3 b_2^{-1} b_3^{-1} b_1 a_2^{-1} \rangle = \langle a_2 a_1 a_2^{-1} a_1^2 a_2^{-1}, b_2 b_1^{-1} b_2^2 \rangle \\ &= \langle \psi(-15 + 10i + 2j + 20k), \psi(-11 - 10i - 2j - 20k) \rangle \cong \mathbb{Z}^2 ,\end{aligned}$$

which is not a period subgroup since $n(\Gamma_0) = 126$, $\left(\frac{-126}{5}\right) = 1$ and $\left(\frac{-126}{3}\right) = 0$.

One could conjecture that every maximal abelian subgroup of Γ is conjugate to either a period subgroup or to a subgroup of Γ_p or Γ_l. The next example shows that this conjecture is not true. We need the following definition and Lemma 3.11:

If $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$, let $m(x) = |x|^2 - \Re(x)^2 = x_1^2 + x_2^2 + x_3^2$, where $\Re(x) = x_0$ denotes the real part of x. Observe that $m(x) = \lambda^2 n(x)$ for some integer λ.

Lemma 3.11. *Let $x, y \in \mathbb{H}(\mathbb{Z})$, then $m(xy\overline{x}) = (|x|^2)^2 m(y)$.*

Proof. Using the rules $\Re(xy) = \Re(yx)$ and $|xy|^2 = |x|^2|y|^2$, we conclude

$$m(xy\overline{x}) = |xy\overline{x}|^2 - \Re(xy\overline{x})^2 = (|x|^2)^2|y|^2 - (|x|^2\Re(y))^2 = (|x|^2)^2 m(y) .$$

□

Example 3.12. Let $\Gamma = \Gamma_{3,5}$ and $a_2 b_3 = \psi(3 + 2i + j + k)$. The group $\Gamma_0 = Z_\Gamma(a_2 b_3)$ is a maximal abelian subgroup of Γ such that $n(\Gamma_0) = 6$. We fix any element $\gamma = \psi(x) \in \Gamma$.

The maximal abelian subgroup $\gamma\Gamma_0\gamma^{-1}$ is not a subgroup of Γ_3 or Γ_5, since $\gamma a_2 b_3 \gamma^{-1} \in \gamma\Gamma_0\gamma^{-1}$ is the ψ-image of $x(3 + 2i + j + k)\overline{x}$ whose norm is a product of an odd power of 3 and an odd power of 5.

We claim that $\gamma\Gamma_0\gamma^{-1}$ is not a period subgroup. If $|x|^2 = 3^r 5^s$, $r, s \geq 0$, then by Lemma 3.11

$$(3^r 5^s)^2.6 = m(x(3 + 2i + j + k)\overline{x}) = \lambda^2 n(\gamma\Gamma_0\gamma^{-1})$$

for some integer λ. It follows that $3 \mid n(\gamma\Gamma_0\gamma^{-1})$, in particular

$$\left(\frac{-n(\gamma\Gamma_0\gamma^{-1})}{3}\right) = 0$$

and Proposition 3.6 proves the claim.

Since any maximal abelian subgroup of rank 2 is conjugate to a period subgroup, it also follows that $\Gamma_0 \cong \mathbb{Z}$. See Figure 4 for a periodic vertical strip of width 1 which is globally invariant under the action of $a_2 b_3$. Note that $(a_2 b_3)^2 = b_2 b_3$. Therefore $a_2 b_3$ acts upon the strip by glide reflection and the unique axis of $a_2 b_3$ is the vertical central line of the strip.

It is well-known that period subgroups in Γ always exist. See for example [Rat, Proposition 4.2] for an elementary proof of this fact, using doubly periodic tilings of the Euclidean plane by unit squares. We mention a corollary of this in terms of integer quaternions.

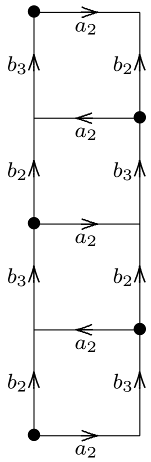

FIGURE 4. Part of a periodic vertical strip.

Corollary 3.13. *Given any pair* (p, l) *of distinct odd primes, there are* $x, y \in \tilde{\Gamma}$ *and* $1 \leq r \leq 4(p+1)^2(l+1)^2$ *such that* $xy = yx$ *and*

$$|x|^2 = p^r, |y|^2 = l^r, \left(\frac{-n(x)}{p}\right) = \left(\frac{-n(y)}{l}\right) = 1\,.$$

The integer r in this corollary comes from the constructive proof of [Rat, Proposition 4.2], and its upper bound is certainly not optimal. In fact, if $p, l \equiv 1 \pmod 4$, there is a direct proof of Corollary 3.13 (with $r = 1$), applying the Two Square Theorem.

References

[BH] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.

[DSV] G. Davidoff, P. Sarnak and A. Valette, *Elementary number theory, group theory, and Ramanujan graphs*, London Mathematical Society Student Texts, 55. Cambridge University Press, Cambridge, 2003.

[Dic] L. E. Dickson, Arithmetic of quaternions, *Proc. London Math. Soc. (2)*, **20** (1922), 225–232.

[Moz] S. Mozes, Actions of Cartan Subgroups, *Israel J. Math.*, **90** (1995), 253–294.

[Lub] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics, 125. Birkhäuser Verlag, Basel, 1994.

[MR] A. G. Myasnikov and V. N. Remeslennikov, Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups, *Internat. J. Algebra Comput.*, **6** (1996), no. 6, 687–711.

[Pra] G. Prasad, Lattices in Semisimple Groups over Local Fields. Studies in algebra and number theory. *Adv. Math. Suppl. Studies*, **6** (1979), 285–356.

[Rat] D. Rattaggi, *Computations in groups acting on a product of trees: normal subgroup structures and quaternion lattices*, Ph.D. thesis, ETH Zürich, 2004.

[Tit] J. Tits, Free subgroups in linear groups, *J. Algebra*, **20** (1972), 250–270.

[Vig] M.-F. Vignéras, *Arithmétique des Algèbres de Quaternions*, Springer-Verlag, Berlin, 1980.

[Wis] D. Wise, Complete square complexes, Preprint September 11, 2003, www.math.mcgill.ca/wise/papers.html

Université de Genève, Section de mathématiques, 2–4 rue du Lièvre, CP 64, CH–1211 Genève 4, Switzerland.
E-mail address: rattaggi@math.unige.ch

Mathematics Department, University of Newcastle, NE1 7RU, U.K.
E-mail address: A.G.Robertson@newcastle.ac.uk