ON DIRECT PRODUCT SUBGROUPS OF $\mathrm{SO}_3(\mathbb{R})$

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Abstract. Let $G_1 \times G_2$ be a subgroup of $\mathrm{SO}_3(\mathbb{R})$ such that the two factors G_1 and G_2 are non-trivial groups. We show that if $G_1 \times G_2$ is not abelian, then one factor is the (abelian) group of order 2, and the other factor is non-abelian and contains an element of order 2. There exist finite and infinite such non-abelian subgroups.

Let F_2 be the free group of rank 2. It is well-known that the group $\mathrm{SO}_3(\mathbb{R})$ has subgroups isomorphic to F_2, e.g.

$$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}, \begin{pmatrix} -3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & -3/5 \end{pmatrix} \right\rangle_{\mathrm{SO}_3(\mathbb{R})} \cong F_2,$$

and subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$, like

$$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -15/17 & -8/17 \\ 0 & 8/17 & -15/17 \end{pmatrix} \right\rangle_{\mathrm{SO}_3(\mathbb{R})} \cong \mathbb{Z} \times \mathbb{Z}.$$

However, $\mathrm{SO}_3(\mathbb{R})$ has no subgroups isomorphic to $\mathbb{Z} \times F_2$. More precisely, if $G_1 \times G_2$ is a non-abelian subgroup of $\mathrm{SO}_3(\mathbb{R})$ such that G_1, G_2 are non-trivial, then G_1, G_2 both contain an element of order 2, and moreover G_1 or G_2 is abelian. We will give an elementary proof of these results (Proposition 7 and Proposition 14) using the Hamilton quaternion algebra $\mathbb{H}(\mathbb{R})$. Additionally, we will show in Proposition 16 that any non-trivial element in the abelian factor has order 2 and in Theorem 18 that in fact the abelian factor is the group of order 2.

Recall that elements $x \in \mathbb{H}(\mathbb{R})$ are of the form $x = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$, and multiplication in $\mathbb{H}(\mathbb{R})$ is induced by the rules $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. The *norm* of x is by definition $|x|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 \in \mathbb{R}$. We say that $x, y \in \mathbb{H}(\mathbb{R})$ are *perpendicular* (denoted by $x \perp y$), if $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$ (i.e. if $(x_1, x_2, x_3)^T$, $(y_1, y_2, y_3)^T$ are perpendicular as vectors in $\mathbb{R}^3$). There is a surjective homomorphism ϑ from the multiplicative group $\mathbb{H}(\mathbb{R}) \setminus \{0\}$ to $\mathrm{SO}_3(\mathbb{R})$ defined by

$$x \mapsto \frac{1}{|x|^2} \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}.$$

It is easy to check that

$$\ker(\vartheta) = Z(\mathbb{H}(\mathbb{R}) \setminus \{0\}) = \{x \in \mathbb{H}(\mathbb{R}) \setminus \{0\} : x = x_0\}$$

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which we will identify with $\mathbb{R} \setminus \{0\}$. Note that if $x \in \mathbb{H}(\mathbb{R}) \setminus \mathbb{R}$, then the axis of the rotation $\vartheta(x)$ is the line through $(0,0,0)^T$ and $(x_1, x_2, x_3)^T$ in $\mathbb{R}^3$. Next, we prove three basic lemmas about (anti-)commutation of quaternions.

Lemma 1. *Let $x, y \in \mathbb{H}(\mathbb{R}) \setminus \{0\}$. Then $xy = -yx$, if and only if $x_0 = y_0 = 0$ and $x \perp y$.*

Proof. Only using quaternion multiplication, we get $xy = -yx$ if and only if the following four equations hold:

$$\begin{aligned} x_1y_1 + x_2y_2 + x_3y_3 &= x_0y_0 \\ x_0y_1 + x_1y_0 &= 0 \\ x_0y_2 + x_2y_0 &= 0 \\ x_0y_3 + x_3y_0 &= 0. \end{aligned}$$

Thus if $x_0 = y_0 = 0$ and $x \perp y$, then clearly $xy = -yx$.

To prove the converse, suppose that $xy = -yx$ and (by contradiction) $x_0 \neq 0$. Then from the four equations, we have $x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 = 0$ and

$$y_1 = \frac{-x_1y_0}{x_0}, \quad y_2 = \frac{-x_2y_0}{x_0}, \quad y_3 = \frac{-x_3y_0}{x_0}.$$

It follows that

$$x_0y_0 + \frac{x_1^2y_0}{x_0} + \frac{x_2^2y_0}{x_0} + \frac{x_3^2y_0}{x_0} = 0$$

and therefore $y_0|x|^2 = 0$. Since $|x|^2 \geq x_0^2 > 0$, we get $y_0 = 0$ which implies $y_1 = 0$, $y_2 = 0$ and $y_3 = 0$, hence the contradiction $y = 0$, and we conclude $x_0 = 0$. The four original equations become $x_1y_1 + x_2y_2 + x_3y_3 = 0$ (i.e. $x \perp y$ as required) and $x_1y_0 = 0$, $x_2y_0 = 0$, $x_3y_0 = 0$, which implies $y_0 = 0$ (using $x \neq 0$) and we are done. □

Lemma 2. *Two quaternions $x, y \in \mathbb{H}(\mathbb{R})$ commute, if and only if $(x_1, x_2, x_3)^T$ and $(y_1, y_2, y_3)^T$ are linearly dependent over $\mathbb{R}$.*

Proof. This follows from the computation

$$\begin{aligned} xy - yx &= 2(x_2y_3 - x_3y_2)i + 2(x_3y_1 - x_1y_3)j + 2(x_1y_2 - x_2y_1)k \\ &= 2\begin{vmatrix} i & x_1 & y_1 \\ j & x_2 & y_2 \\ k & x_3 & y_3 \end{vmatrix}. \end{aligned}$$

□

Lemma 3. *Let $x, y, z \in \mathbb{H}(\mathbb{R}) \setminus \mathbb{R}$. If $xy = yx$ and $xz = zx$, then $yz = zy$. In other words, the group $\mathbb{H}(\mathbb{R}) \setminus \{0\}$ is commutative transitive on non-central elements.*

Proof. By assumption we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The statement follows now directly from Lemma 2. □

To describe the structure of direct product subgroups of $\mathrm{SO}_3(\mathbb{R})$, we give some general definitions.

Definition 4. We call a direct product $G_1 \times G_2$ *non-trivial*, if both G_1 and G_2 are non-trivial groups.

Definition 5. We say that the group G satisfies property

- (P_1), if G is abelian.
- (P_2), if G is CSA, i.e. if all its maximal abelian subgroups are malnormal (in other words, if for any maximal abelian subgroup $H < G$ and any $g \in G \setminus H$ the intersection of gHg^{-1} with H is trivial).
- (P_3), if G is commutative transitive, i.e. if $xy = yx$, $xz = zx$ always implies $yz = zy$ (provided $x, y, z \in G \setminus \{1\}$).
- (P_4), if any non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian (equivalently, if in any non-trivial direct product subgroup $G_1 \times G_2 < G$ both factors G_1, G_2 are abelian).
- (P_5), if any non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian, or exactly one factor is the abelian group of order 2 and the other factor is a non-abelian group containing an element of order 2.
- (P_6), if any non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian, or exactly one factor is abelian such that the non-abelian factor contains an element of order 2 and any non-trivial element in the abelian factor has order 2.
- (P_7), if any non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian or both factors G_1, G_2 contain an element of order 2.
- (P_8), if any torsion-free non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian.
- (P_9), if G contains no subgroup $\mathbb{Z} \times F_2$.
- (P_{10}), if G contains no subgroup $F_2 \times F_2$.
- (R_3), if G is commutative transitive on non-central elements, i.e. if $xy = yx$, $xz = zx$ always implies $yz = zy$ (provided $x, y, z \in G \setminus ZG$).
- (R_4), if any non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian, or one factor is non-abelian and the other factor is contained in the center of G.
- (R_6), if in any non-trivial direct product subgroup $G_1 \times G_2 < G$ at least one factor is abelian.

Remark 6. The arrows in the following diagram stand for implications. For example "$(P_1) \longrightarrow (P_2)$" means "if a group G satisfies property (P_1), then G satisfies property (P_2)". These implications follow directly from the given definitions, except maybe $(P_2) \longrightarrow (P_3)$ which is also easy to prove, see [1, Proposition 7].

$$\begin{array}{ccccccccccccc}
(P_2) & \longrightarrow & (P_3) & \longrightarrow & (P_4) & \longrightarrow & (P_5) & \longrightarrow & (P_6) & \longrightarrow & (P_7) & \longrightarrow & (P_8) \\
\uparrow & & \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\
(P_1) & & (R_3) & \longrightarrow & (R_4) & & \longrightarrow & & (R_6) & \longrightarrow & (P_{10}) & \longleftarrow & (P_9)
\end{array}$$

We will show in Proposition 7 that $\mathrm{SO}_3(\mathbb{R})$ satisfies property (P_7), and in Proposition 14 that $\mathrm{SO}_3(\mathbb{R})$ satisfies property (R_6), using the map ϑ and our lemmas on quaternions. These results will be refined in Proposition 16 and Theorem 18 to prove that $\mathrm{SO}_3(\mathbb{R})$ satisfies property (P_6) and (P_5).

For a group with trivial center, e.g. for $\mathrm{SO}_3(\mathbb{R})$, properties (P_4) and (R_4) are equivalent. In Observation 13, we illustrate by two examples that $\mathrm{SO}_3(\mathbb{R})$ does not satisfy property (P_4) (and hence does not satisfy property (R_4)). As a preparation, Observation 11 shows that $\mathrm{SO}_3(\mathbb{R})$ does not satisfy property (P_3).

Proposition 7. *The group* $\mathrm{SO}_3(\mathbb{R})$ *satisfies property* (P_7).

Proof. Let $G_1 \times G_2$ be a non-trivial direct product subgroup of $\mathrm{SO}_3(\mathbb{R})$ and suppose that G_1 or G_2 does not contain an element of order 2. We have to prove that $G_1 \times G_2$ is abelian. Let E be the identity matrix in $\mathrm{SO}_3(\mathbb{R})$, and take any $A \in G_1 \setminus \{E\}$, $B, C \in G_2 \setminus \{E\}$. Then $AB = BA$ and $AC = CA$. Take any $x, y, z \in \mathbb{H}(\mathbb{R}) \setminus \mathbb{R}$ such that $\vartheta(x) = A$, $\vartheta(y) = B$ and $\vartheta(z) = C$. We have $\vartheta(x)\vartheta(y) = \vartheta(y)\vartheta(x)$, hence $xyx^{-1}y^{-1} \in \ker(\vartheta)$, i.e. $xy = \lambda yx$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Taking the norm, and using the rule $|xy|^2 = |x|^2|y|^2$, we see that $\lambda \in \{-1, 1\}$, in other words $xy = yx$ or $xy = -yx$. Similarly, $AC = CA$ implies that $xz = zx$ or $xz = -zx$.

In the case $xy = -yx$, we get $x_0 = y_0 = 0$ by Lemma 1. But then $x^2, y^2 \in \mathbb{R} \setminus \{0\}$ and $A^2 = \vartheta(x^2) = E$, $B^2 = \vartheta(y^2) = E$, hence both G_1 and G_2 contain an element of order 2, a contradiction to our assumption. In the same way, if $xz = -zx$, then we get the contradiction $A^2 = C^2 = E$.

Hence we always have $xy = yx$ and $xz = zx$. Using Lemma 3, we get $yz = zy$ and therefore $BC = CB$. This shows that G_2 is abelian. Similarly, taking two matrices in $G_1 \setminus \{E\}$ and one matrix in $G_2 \setminus \{E\}$, one shows that G_1 is abelian. □

Corollary 8. *The group* $\mathrm{SO}_3(\mathbb{R})$ *contains no subgroup* $\mathbb{Z} \times F_2$ *and no subgroup* $F_2 \times F_2$.

Proof. Property (P_7) implies property (P_9) and (P_{10}). □

Remark 9. A group is called *coherent* if every finitely generated subgroup is finitely presented. Any group containing a subgroup $F_2 \times F_2$ is incoherent. Therefore the non-existence of subgroups $F_2 \times F_2$ is a necessary condition for coherence, although it is not a sufficient condition since there are for example incoherent (hyperbolic) groups (using [2]) not containing $\mathbb{Z} \times F_2$ subgroups. It is a question of Serre ([3, p.734]) whether $\mathrm{GL}_3(\mathbb{Q})$ is coherent.

Question 10. *Is* $\mathrm{SO}_3(\mathbb{R})$ *coherent?*

Using the idea of the proof of Proposition 7, we see that any subgroup of $\mathrm{SO}_3(\mathbb{R})$ which does not contain elements of order 2 (in particular any torsion-free subgroup of $\mathrm{SO}_3(\mathbb{R})$) is commutative transitive. However $\mathrm{SO}_3(\mathbb{R})$ itself is not commutative transitive:

Observation 11. *The group* $\mathrm{SO}_3(\mathbb{R})$ *does not satisfy property* (P_3).

This observation will directly follow from Observation 13, but we give a short alternative proof here.

Proof. Take

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ C := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

then $AB = BA$ and $AC = CA$, but $BC \neq CB$.

Note that $A = \vartheta(i)$, $B = \vartheta(1 + i)$, $C = \vartheta(j)$ and $i(i + 1) = (i + 1)i$, $ij = -ji$, $(i + 1)j \neq \pm j(i + 1)$. □

Corollary 12. *There is a group* G *which is commutative transitive on non-central elements, but such that* $G/Z(G)$ *is not commutative transitive on non-central elements (and therefore such that* $G/Z(G)$ *is not commutative transitive).*

Proof. Take $G = \mathbb{H}(\mathbb{R}) \setminus \{0\}$ such that $G/ZG \cong \mathrm{SO}_3(\mathbb{R})$ and note that $Z(\mathrm{SO}_3(\mathbb{R}))$ is the trivial group. □

The matrices A, B, C from the proof of Observation 11 generate a non-abelian subgroup $\langle A, B, C\rangle$ of $\mathrm{SO}_3(\mathbb{R})$. However, this group cannot be used to prove that $\mathrm{SO}_3(\mathbb{R})$ does not satisfy property (P_4), since $A = B^2$ and $\langle A, B, C\rangle = \langle B, C\rangle$ is the dihedral group of order 8 which is not decomposable as a non-trivial direct product. Nevertheless, there *are* non-abelian non-trivial direct product subgroups of $\mathrm{SO}_3(\mathbb{R})$.

Observation 13. *The group* $\mathrm{SO}_3(\mathbb{R})$ *does not satisfy property* (P_4).

Proof. We give two examples of a non-abelian non-trivial direct product subgroup of $\mathrm{SO}_3(\mathbb{R})$, at first an infinite example.

Let $A = \vartheta(i)$, $C = \vartheta(j)$ as in the proof of Observation 11 and let

$$\tilde{B} := \vartheta(1 + 2i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}.$$

We claim that $\langle A, \tilde{B}, C\rangle$ is a non-abelian non-trivial direct product subgroup of $\mathrm{SO}_3(\mathbb{R})$.

First we want to show by contradiction that $A \notin \langle \tilde{B}, C\rangle$. Since $C\tilde{B} = \tilde{B}^{-1}C$ and $C\tilde{B}^{-1} = \tilde{B}C$, any word in the letters $\tilde{B}$, $\tilde{B}^{-1}$, $C = C^{-1}$ can be brought to the form $\tilde{B}^nC$ or $\tilde{B}^n$ for some $n \in \mathbb{Z}$. If we suppose that $A \in \langle \tilde{B}, C\rangle$, then, looking at the upper left entry (which is 1 in A and $\tilde{B}$, but -1 in C), we see that A cannot be written as $\tilde{B}^nC$ and therefore $A = \tilde{B}^n$ for some $n \in \mathbb{Z} \setminus \{0\}$. But since A has order 2, we get $\tilde{B}^{2n} = E$, which contradicts the fact that $\tilde{B}$ has infinite order.

Since $\langle A\rangle$ has only two elements and $A \notin \langle \tilde{B}, C\rangle$, we get $\langle A\rangle \cap \langle \tilde{B}, C\rangle = \{E\}$. Moreover, it is easy to check that A commutes with $\tilde{B}$ and with C. Therefore $\langle A, \tilde{B}, C\rangle < \mathrm{SO}_3(\mathbb{R})$ is a direct product of the group $\langle A\rangle$ of order 2 and the (infinite) non-abelian (solvable) group $\langle \tilde{B}, C\rangle$.

As a finite example we can take the dihedral group of order 12, generated for example by the two matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This group is isomorphic to a direct product of the (non-abelian) dihedral group of order 6 (which is isomorphic to the symmetric group S_3) and the group of order 2. □

Proposition 14. *The group* $\mathrm{SO}_3(\mathbb{R})$ *satisfies property* (R_6).

Proof. Suppose by contradiction that $G_1 \times G_2$ is a non-trivial direct subgroup of $\mathrm{SO}_3(\mathbb{R})$ such that G_1 and G_2 are non-abelian. First take $A, B \in G_1 \setminus \{E\}$ such that $AB \neq BA$ and $C, D \in G_2 \setminus \{E\}$ such that $CD \neq DC$. Now take $x, y, z, w \in \mathbb{H}(\mathbb{R}) \setminus \mathbb{R}$ such that $\vartheta(x) = A$, $\vartheta(y) = B$, $\vartheta(z) = C$, $\vartheta(w) = D$. Then we have $xy \neq \pm yx$, $zw \neq \pm wz$ and (by the same argument as in the proof of Proposition 7) $xz = \pm zx$, $xw = \pm wx$, $yz = \pm zy$, $yw = \pm wy$.

Suppose that $xz = zx$. If $xw = wx$ then we get by Lemma 3 the contradiction $zw = wz$, hence $xw = -wx$. But then by Lemma 1, $w_0 = 0$ and $x \perp w$. Since

$(x_1, x_2, x_3)^T$ and $(z_1, z_2, z_3)^T$ are linearly dependent by Lemma 2, we conclude $z \perp w$. Since $wz \neq -zw$, we have $z_0 \neq 0$ by Lemma 1, hence $yz = zy$ again by Lemma 1, and $xy = yx$ by Lemma 3, a contradiction.

We have shown that $xz = -zx$. Similarly, it follows that $xw = -wx$, $yz = -zy$ and $yw = -wy$. Lemma 1 implies $x \perp z$ and $x \perp w$. Since $zw \neq wz$, z and w are linearly independent by Lemma 2 and span the plane perpendicular to x. We also have $y \perp z$ and $y \perp w$ by Lemma 1, hence x and y are linearly dependent and we get the contradiction $xy = yx$ by Lemma 2. □

Lemma 15. *Let $A \in \mathrm{SO}_3(\mathbb{R})$ be a rotation of order at least 3. Then the centralizer of A in $\mathrm{SO}_3(\mathbb{R})$ consists of all rotations about the axis of A.*

Proof. Without loss of generality, we may assume that A is a rotation of order at least 3 about the x-axis, hence

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix},$$

such that $\sin\phi \neq 0$. Suppose that the matrix

$$B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix} \in \mathrm{SO}_3(\mathbb{R})$$

commutes with A. Then $AB = BA$ gives the conditions

$$b_9 \sin\phi = b_5 \sin\phi$$
$$-b_8 \sin\phi = b_6 \sin\phi$$

and

$$b_2(1 - \cos\phi) = b_3 \sin\phi$$
$$-b_3(1 - \cos\phi) = b_2 \sin\phi$$
$$-b_4(1 - \cos\phi) = b_7 \sin\phi$$
$$b_7(1 - \cos\phi) = b_4 \sin\phi.$$

The first two equations imply $b_5 = b_9$ and $b_6 = -b_8$. The third and fourth equation imply

$$b_2 = \frac{-b_3(1 - \cos\phi)}{\sin\phi} \quad \text{and} \quad \frac{-b_3(1 - \cos\phi)^2}{\sin\phi} = b_3 \sin\phi,$$

hence

$$-b_3(1 - 2\cos\phi) = b_3(\sin^2\phi + \cos^2\phi) = b_3.$$

If $b_3 \neq 0$ then $1 - 2\cos\phi = -1$, hence $\cos\phi = 1$ and we get the contradiction $\sin\phi = 0$. Thus $b_3 = 0$ and $b_2 = 0$. Similarly, the fifth and sixth equation lead to $b_4 = b_7 = 0$, hence

$$B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_5 & -b_8 \\ 0 & b_8 & b_5 \end{pmatrix}$$

We exclude the case $b_1 = -1$ computing the determinant of B, and conclude

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}$$

for some ψ. □

Proposition 16. *The group* $\mathrm{SO}_3(\mathbb{R})$ *satisfies property* (P_6).

Proof. Let $G_1 \times G_2$ be a subgroup of $\mathrm{SO}_3(\mathbb{R})$ such that G_2 is non-abelian and G_1 is abelian and non-trivial. Using Proposition 7 and Proposition 14, it remains to prove that any non-trivial element of G_1 has order 2. Therefore suppose that $A \in G_1 \setminus \{E\}$ has order at least 3. Then by Lemma 15, any element in G_2 is a rotation about the axis of A, which contradicts our assumption that G_2 is non-abelian. □

Lemma 17. *The two matrices*

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & \sin\phi & -\cos\phi \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & \sin\psi & -\cos\psi \end{pmatrix} \in \mathrm{SO}_3(\mathbb{R})$$

commute, if and only if

$$\frac{\phi}{2} - \frac{\psi}{2} \in \{k \cdot \frac{\pi}{2} : k \in \mathbb{Z}\}.$$

In particular, these two $180°$*-rotations commute, if and only if their axes (which lie in the* yz*-plane) are identical or perpendicular.*

Proof. Matrix multiplication gives the condition $\sin\phi \cdot \cos\psi = \cos\phi \cdot \sin\psi$, hence

$$0 = \sin\phi \cdot \cos\psi - \cos\phi \cdot \sin\psi = \sin(\phi - \psi)$$

and

$$\phi - \psi \in \{k \cdot \pi : k \in \mathbb{Z}\}.$$

□

Theorem 18. *The group* $\mathrm{SO}_3(\mathbb{R})$ *satisfies property* (P_5).

Proof. Let $G_1 \times G_2$ be a subgroup of $\mathrm{SO}_3(\mathbb{R})$ such that G_2 is non-abelian and G_1 is abelian and non-trivial. Applying Proposition 16, it remains to show that G_1 has order 2. Let $A \in G_1 \setminus \{E\}$. Without loss of generality we may assume that A is a rotation about the x-axis. It has order 2 by Proposition 16, hence

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Any element in $G_1 \setminus \{E\}$ has order 2 and commutes with A. An easy computation shows that if an element in $\mathrm{SO}_3(\mathbb{R})$ commutes with A, then it has either the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & \sin\phi & -\cos\phi \end{pmatrix},$$

i.e. it is either a rotation about the x-axis, or a rotation about an axis in the yz-plane by an angle of $180°$. The only element of order 2 of the first form is A itself. Hence if $G_1 \setminus \{E, A\}$ is not empty, then it contains only elements of the second form. Since G_1 is abelian, $G_1 \setminus \{E, A\}$ contains by Lemma 17 at most two elements, and G_1 has therefore at most 4 elements. However, we know by Proposition 7 that also G_2 contains an element of order 2 commuting with A, hence G_1 has less than 4 elements. Since $A \in G_1$ has order 2, we conclude that G_1 has exactly 2 elements. □

Remark 19. All statements in this article remain true if we replace $\mathbb{R}$ by $\mathbb{Q}$. The only construction where we have used irrational numbers was in the second part of Observation 13.

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